# Automating Natural Deduction 1

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# Overview

1. Low-Level Proof Scripts
2. Natural Deduction Rules for Propositional Calculus
3. Automation with the Classical Reasoner

# Outline

# Motivation

- The simp tactic uses equations ($s = t$) to rewrite and simplify a goal.
- Proof by simplification is the gold standard for efficiency and usability.
- But it's not always possible to prove a goal with just the simplifier.
- Isabelle provides tactics for automating natural deduction.
- One of the core reasoning techniques, alongside the simplifier.
- You need to understand the use of low-level proof scripts.

**Proof Script**

```
lemma "P ⟹ (P ∧ (P ∨ Q))"
  apply (rule conjI)
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  apply (rule disjI1)
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# Proof Scripts and Tactics

- Low-level proof language: like the "machine code" for proofs
- Proof script: sequence of commands acting on a proof state
- Proof state: subgoals
- Subgoals: collection of hypotheses and outstanding conclusions
- Commands invoke proof tactics to decompose and eliminate the goals

- Proof script uses the keyword to continue the proof
- Tactics can fail if the subgoal is in the wrong form
- Proof is complete once all subgoals are eliminated
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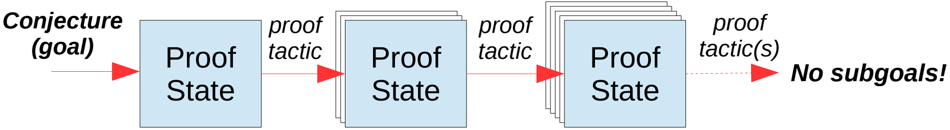

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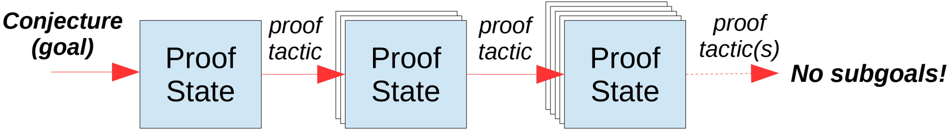


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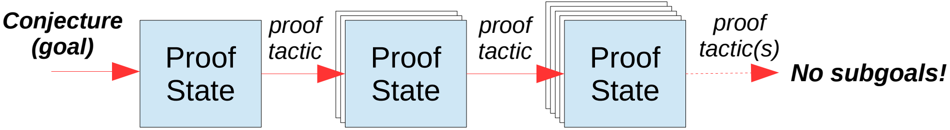


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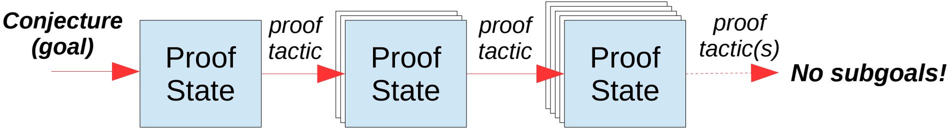


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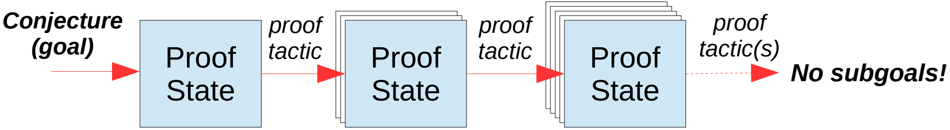


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- The proof state consists of $n$ subgoals.
- The aim is to discharge (remove by proof) all remaining subgoals.
- Subgoals have the form $[\![P_1; \ldots; P_n]\!] \Longrightarrow Q$.
- $P_1 \ldots P_n$ is a set of hypotheses – can be used to prove the subgoal.
- $Q$ is the conclusion of the subgoal, the thing to be proved.
- `assumes P1 ... Pn shows Q` creates a single initial subgoal:
  - $[\![P_1; \ldots; P_n]\!] \Longrightarrow Q$
- Tactics often act on the first subgoal, but some act on all (`simp_all`).
- We can manipulate a subgoal in several ways:
  - Split into several additional subgoals by introduction or elimination.
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## Natural Deduction Rules

$$\frac{\Gamma \vdash P_1 \quad \Gamma \vdash P_2 \cdots \Gamma \vdash P_n}{\Gamma \vdash Q}$$

- Meaning: a proof of the conclusion $Q$ follows from premises $P_1 \cdots P_n$.
- $\Gamma \vdash Q$ is a sequent: valid when $Q$ holds subject to the hypotheses in $\Gamma$.

## Example (Syllogism)

$$\frac{\text{Socrates is a man} \qquad \text{All men are mortal}}{\text{Socrates is a mortal}}$$

Two main kinds of deduction rules:

1. Introduction rules (rule tactic): backwards reasoning.
2. Elimination rules (erule tactic): forwards reasoning.

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2. Elimination rules (erule tactic): forwards reasoning.

# Natural Deduction

## Natural Deduction Rules

$$\frac{\Gamma \vdash P_1 \quad \Gamma \vdash P_2 \cdots \Gamma \vdash P_n}{\Gamma \vdash Q}$$

- Meaning: a proof of the conclusion $Q$ follows from premises $P_1 \cdots P_n$.
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## Example (Syllogism)

$$\frac{\text{Socrates is a man} \qquad \text{All men are mortal}}{\text{Socrates is a mortal}}$$

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conjI TrueI

disjI1 disjI2

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# Example: Proof by Introduction

## Natural Deduction Proof

asm

asm disjI1

conjI

## Proof Script

```
lemma "P ⟹ (P ∧ (P ∨ Q))"
  apply (rule conjI)
  (* Subgoal 1 *)
  apply assumption
  (* Subgoal 2 *)
  apply (rule disjI1)
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  done
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# Example: Proof by Introduction

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$$\dfrac{\dfrac{}{P \vdash P}\,\texttt{asm} \qquad \dfrac{\dfrac{}{P \vdash P}\,\texttt{asm}}{P \vdash P \vee Q}\,\texttt{disjI1}}{P \vdash P \wedge (P \vee Q)}\,\texttt{conjI}$$

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## Proof State

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  done
```

# Example: Proof by Introduction (Again)

**Proof State**

1. $P \Longrightarrow (P \wedge (P \vee Q))$.

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# Example: Proof by Introduction (Again)

## Proof State

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2. $P \Longrightarrow P \vee Q$.

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**Proof State**

No subgoals!

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```

# More Introduction Rules

### Introduction Rules

impI conjI refl

```
lemma "P ⟶ (P ∧ (True = True))"
apply (rule impI)  (* P ⟹ (P ∧ (True = True)) *)
apply (rule conjI) (* P ⟹ P and P ⟹ (True = True)) *)
apply (assumption) (* P ⟹ (True = True)) *)
apply (rule refl)
done
```

# More Introduction Rules

**Introduction Rules**

$$\frac{P, \Gamma \vdash Q}{\Gamma \vdash P \longrightarrow Q}\ \texttt{impI} \qquad \frac{\Gamma \vdash P \longrightarrow Q \qquad \Gamma \vdash Q \longrightarrow P}{\Gamma \vdash P \longleftrightarrow Q}\ \texttt{iffI} \qquad \frac{-}{\Gamma \vdash t = t}\ \texttt{refl}$$

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lemma "P ⟶ (P ∧ (True = True))"
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lemma "P ⟶ (P ∧ (True = True))"
apply (rule impI)   (* P ⟹ (P ∧ (True = True)) *)
apply (rule conjI) (* P ⟹ P and P ⟹ (True = True)) *)
apply (assumption) (* P ⟹ (True = True)) *)
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# More Introduction Rules

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# Elimination Rules

- What can we deduce from hypothesis $P$? Refine a subgoal.

## Elimination Rules for Propositional Calculus

- Elimination rules use erule tactical (`erule conjE`)
- Find first hypothesis matching first premise of elimination rule
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# Elimination Rules

- What can we deduce from hypothesis $P_i$? Refine a subgoal.

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# Example: Proof by Elimination (1)

How do we prove $P \wedge (P \longrightarrow Q) \longrightarrow Q$?

## Natural Deduction Proof

asm asm

impE

conjE

impI

## Proof Script

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lemma "P ∧ (P ⟶ Q) ⟶ Q"
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$P \Longrightarrow P$

asm

$[\![P; Q]\!] \Longrightarrow Q$

impE

$[\![(P \longrightarrow Q); P]\!] \Longrightarrow Q$

conjE

$P \wedge (P \longrightarrow Q) \Longrightarrow Q$

impI

$P \wedge (P \longrightarrow Q) \longrightarrow Q$

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# Example: Proof by Elimination (1)

How do we prove $P \wedge (P \longrightarrow Q) \longrightarrow Q$?

## Natural Deduction Proof

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{P \Longrightarrow P}\,\texttt{asm} \qquad \dfrac{}{[\![P;\; Q]\!] \Longrightarrow Q}\,\texttt{asm}}{[\![(P \longrightarrow Q);\; P]\!] \Longrightarrow Q}\,\texttt{impE}}{P \wedge (P \longrightarrow Q) \Longrightarrow Q}\,\texttt{conjE}}{P \wedge (P \longrightarrow Q) \longrightarrow Q}\,\texttt{impI}
$$

## Proof Script

```
lemma "P ∧ (P ⟶ Q) ⟶ Q"
  apply (rule impI)
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    (* Subgoal 1 *)
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  }{
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  }\,\text{impE}
}{
  \dfrac{P \wedge (P \longrightarrow Q) \Longrightarrow Q}{P \wedge (P \longrightarrow Q) \longrightarrow Q}\,\text{impI}
}\,\text{conjE}
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# Example: Proof by Elimination (2): $(P \wedge P) \longleftrightarrow P$

Proof Script

```
lemma "(P ∧ P) ⟷ P"
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$$(P \wedge P) \longleftrightarrow P$$

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$$\frac{}{(P \wedge P) \longleftrightarrow P}\ \mathtt{iffI}$$

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# Outline

# Natural Deduction in Isar

```
lemma "P ∧ (P ⟶ Q) ⟶ Q"
proof (rule impI)
  assume "P ∧ (P ⟶ Q)"
  hence "P ⟶ Q" "P"
    by (erule_tac conjE, simp_all)
  thus "Q"
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qed
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- Benefit of being readable without seeing the proof state
- A little too verbose for simple predicates

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# Automating Natural Deduction

- Isn't all this low level deduction too much work?
- Fortunately, Isabelle automates natural deduction using the blast tactic.
- Tableaux prover: represent proof tree structure.

**Example**

```
lemma "P ⟶ (P ∧ (P ∨ Q))" by blast
lemma "P ∧ (P ⟶ Q) ⟶ Q" by blast
```

- Blast searches proof tree using introduction and elimination rules.
- `blast intro: thms elim: thms`, or `[intro]`, `[elim]`, and `[dest]`.
- Safe rules marked with "!", e.g. `[intro!]`.
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# Combining Deduction and Simplification with Auto

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- `auto` is a powerful tactic combining deduction (e.g., `blast`) and `simp`.
- Not one-shot: applies safe rules repeatedly, simplifying if possible.
- Won't lose information as long as the specified safe rules are truly safe.
- General technique for breaking a complex goal into several parts.

```
lemma "(x::nat) = y² ⟶ x ≥ y ∧ x ≥ 0"
  by (auto simp add: power2_nat_le_imp_le)
```

- There are also one-shot variants: `force` and `fastforce`.
- `auto` can produce unpredictable and bizarre results.
- Still need manual deduction to understand why automated proof fails.

# Combining Deduction and Simplification with Auto

- `auto` is a powerful tactic combining deduction (e.g., `blast`) and `simp`.
- Not one-shot: applies safe rules repeatedly, simplifying if possible.
- Won't lose information as long as the specified safe rules are truly safe.
- General technique for breaking a complex goal into several parts.

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lemma "(x::nat) = y² ⟶ x ≥ y ∧ x ≥ 0"
  by (auto simp add: power2_nat_le_imp_le)
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- There are also one-shot variants: `force` and `fastforce`.
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- Still need manual deduction to understand why automated proof fails.

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# Applications

- We looked at automating natural deduction in the propositional calculus.
- We demonstrated a far more general technique.
- We define bespoke logics in Isabelle and then reason about them.
- In particular, we can program verification (e.g., Hoare logic).
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# Conclusion

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- Natural deduction in Isabelle/HOL.
- The classical reasoner.

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- Predicate calculus (quantifiers etc.).

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