

Automating Natural Deduction 1

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Overview

- 1 Low-Level Proof Scripts
- 2 Natural Deduction Rules for Propositional Calculus
- 3 Automation with the Classical Reasoner



Outline

- 1 Low-Level Proof Scripts
- 2 Natural Deduction Rules for Propositional Calculus
- 3 Automation with the Classical Reasoner

Motivation

- The **simp** tactic uses equations ($s = t$) to rewrite and simplify a goal.
- Proof by simplification is the **gold standard** for efficiency and usability.
- But it's not always possible to prove a goal with just the simplifier.
- Isabelle provides tactics for automating **natural deduction**.
- One of the **core reasoning techniques**, alongside the simplifier.
- You need to understand the use of low-level **proof scripts**.

Proof Script

```
lemma "P  $\implies$  (P  $\wedge$  (P  $\vee$  Q))"  
  apply (rule conjI)  
  apply assumption  
  apply (rule disjI1)  
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  done
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Proof Scripts and Tactics

- Low-level proof languages are the “machine code” for Isar.
- Proof script: sequence of commands acting on a proof state.
- Proof state: structure containing:
 - Current collection of hypotheses and outstanding conjectures.
 - Commands make proof tactics to decompose and refine the goals.

- Proof script uses the `begin` keyword to denote a block.
- Tactics can fill the subject in the wrong form.
- Proof is complete once all subgoals are eliminated.
- Script terminated with `qed` (or `end`).

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- Commands invoke **proof tactics** to decompose and **refine** the goals.

- Proof script uses the **tactics** to decompose the goals.
- Tactics tell how subgoal is in the wrong form.
- Proof is complete once all subgoals are eliminated.
- Some coordinated with **proof** and **done** commands.

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- **Proof state**: **subgoals**.
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- Commands invoke **proof tactics** to decompose and **refine** the goals.

- Proof script uses the **proof** keyword to declare subgoals.
- Proof state follows subgoal is in the wrong form.
- Proof is complete once all subgoals are eliminated.
- Some commands with **done** keyword.

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- **Proof state**: **subgoals**.
- **Subgoals**: collection of hypotheses and outstanding conjectures.
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● Proof script is a sequence of commands to describe a proof.

● Proof state holds the current state of the ongoing proof.

● Subgoals: collection of all conjectures and outstanding goals.

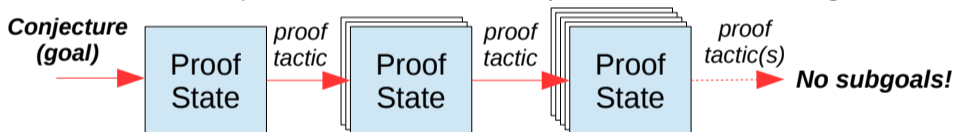
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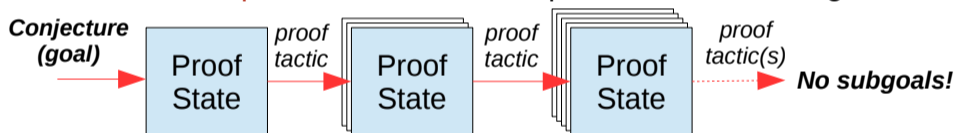
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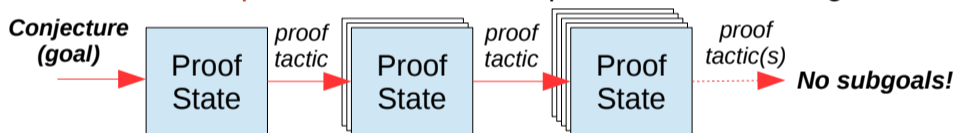
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- Proof script uses the **apply** keyword to execute a tactic.
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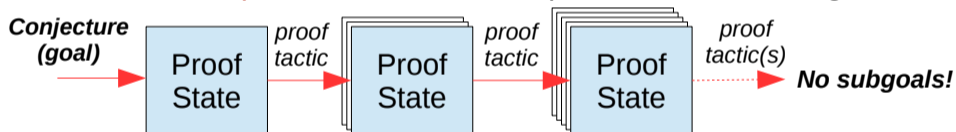
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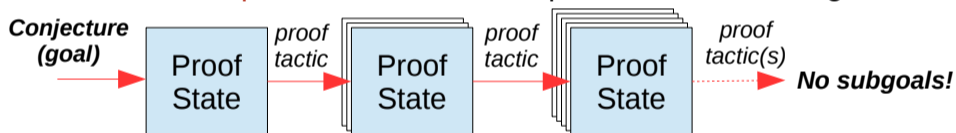
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Proof State

- The proof state consists of a subgoals.
- The aim is to discharge (remove by proof) all remaining subgoals.
- Subgoals have the form $[P_1, \dots, P_n] \Rightarrow Q$.
- P_1, \dots, P_n is a set of hypotheses—can be used to prove the subgoal.
- Q is the conclusion of the subgoal; the thing to be proved.
- $P_1, \dots, P_n, Q \Rightarrow Q$ creates a single initial subgoal.
- $[P_1, \dots, P_n] \Rightarrow Q$
- \vdash has no effect on the first subgoal, but some act on all (simp, refl).
- We can manipulate a subgoal in several ways:
 - Split into several additional subgoals by introduction or elimination.
 - Manipulate the assumptions or the conclusion further ones, e.g. from theorem.
 - Discharge trivial subgoals, e.g. True or $P \Rightarrow P$ using assumption.

Proof State

- The proof state consists of n subgoals.
- The aim is to **discharge** (remove by proof) all remaining subgoals.
- Subgoals have the form $\llbracket P_1; \dots; P_n \rrbracket \implies Q$.
- $P_1 \dots P_n$ is a set of hypotheses – can be used to prove the subgoal.
- Q is the conclusion of the subgoal; the thing to be proved.
- **assumes** $P_1 \dots P_n$ **shows** Q creates a single initial subgoal:
$$\llbracket P_1; \dots; P_n \rrbracket \implies Q$$
- Tactics often act on the **first subgoal**, but some act on **all** (**simp_all**).
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Natural Deduction

Natural Deduction Rules

$$\frac{\Gamma \vdash P_1 \quad \Gamma \vdash P_2 \cdots \Gamma \vdash P_n}{\Gamma \vdash Q}$$

- Meaning: a proof of the conclusion Q follows from premises $P_1 \cdots P_n$.
- $\Gamma \vdash Q$ is a **sequent**: valid when Q holds subject to the hypotheses in Γ .

Example (Syllogism)

$$\frac{\text{Socrates is a man} \quad \text{All men are mortal}}{\text{Socrates is a mortal}}$$

Two main kinds of deduction rules:

- Introduction rules (rule 1-10): backwards reasoning
- Elimination rules (rule 1-10): forwards reasoning

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Natural Deduction Rules

$$\frac{\Gamma \vdash P_1 \quad \Gamma \vdash P_2 \cdots \Gamma \vdash P_n}{\Gamma \vdash Q}$$

- **Meaning:** a proof of the **conclusion** Q follows from **premises** $P_1 \cdots P_n$.
- $\Gamma \vdash Q$ is a **sequent**: valid when Q holds subject to the hypotheses in Γ .

Example (Syllogism)

$$\frac{\text{Socrates is a man} \quad \text{All men are mortal}}{\text{Socrates is a mortal}}$$

Two main kinds of deduction rules:

- **Introducing rules** (rule to get to a specific conclusion)
- **Eliminating rules** (rule to get to a general conclusion)

Natural Deduction

Natural Deduction Rules

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Example (Syllogism)

$$\frac{\text{Socrates is a man} \quad \text{All men are mortal}}{\text{Socrates is a mortal}}$$

Two main kinds of deduction rules:

- 1. **Introducing rules** (rule to add a formula to a context)
- 2. **Eliminating rules** (rule to remove a formula from a context)

Natural Deduction

Natural Deduction Rules

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- 1. **Introduction rules** (rule to add the conclusion to the context)
- 2. **Elimination rules** (rule to remove the conclusion)

Natural Deduction

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Two main kinds of deduction rules:

- 1. **Introduction rules** (e.g. \wedge intro, \vee intro, \rightarrow intro)
- 2. **Elimination rules** (e.g. \wedge elim, \vee elim, \rightarrow elim)

Natural Deduction

Natural Deduction Rules

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Two main kinds of deduction rules:

- 1 Introduction rules (**rule** tactic): backwards reasoning.
- 2 Elimination rules (**erule** tactic): forwards reasoning.

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Natural Deduction Rules

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Introduction Rules

- **Question:** how do we prove the conclusion P ?
- By finding subgoals $P_1 \cdots P_n$, based on current subgoal's conclusion.
- Proof by introduction is often called **backwards reasoning**.

Introduction Rules

- Invoke introduction rules using the **rule** tactic, e.g. **apply**(rule conjI).
- Matches subgoal's conclusion, **copies** any hypotheses to new subgoals.
- Most rules are **safe** introduction rules, but disjI1 and disjI2 aren't.

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Introduction Rules

• `conjI` `disjI1` `disjI2`
• `implI` `exhI` `exhE`
• `exhE` `exhI` `exhE`
• `exhI` `exhE` `exhI`

- Invoke introduction rules using the **rule** tactic, e.g. **apply**(rule conjI).
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Introduction Rules

$$\frac{\Gamma \vdash P \quad \Gamma \vdash Q}{\Gamma \vdash P \wedge Q} \text{ conjI} \qquad \frac{}{\Gamma \vdash \text{True}} \text{ TrueI}$$
$$\frac{\Gamma \vdash P}{\Gamma \vdash P \vee Q} \text{ disjI1} \qquad \frac{\Gamma \vdash Q}{\Gamma \vdash P \vee Q} \text{ disjI2}$$

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Example: Proof by Introduction

Natural Deduction Proof

```
1. P      assumption
2. P ∨ Q  disjI1
3. P ∧ (P ∨ Q) conjI
```

Proof Script

```
lemma "P  $\implies$  (P  $\wedge$  (P  $\vee$  Q))"
  apply (rule conjI)
  (* Subgoal 1 *)
  apply assumption
  (* Subgoal 2 *)
  apply (rule disjI1)
  apply assumption
done
```

Example: Proof by Introduction

Natural Deduction Proof

$$\begin{array}{c} \text{asm} \quad \overline{P} \\ \text{conjI} \quad P \quad P \\ \hline P \vdash P \wedge (P \vee Q) \end{array}$$

Proof Script

```
lemma "P  $\Rightarrow$  (P  $\wedge$  (P  $\vee$  Q))"  
  apply (rule conjI)  
  (* Subgoal 1 *)  
  apply assumption  
  (* Subgoal 2 *)  
  apply (rule disjI1)  
  apply assumption  
done
```

Example: Proof by Introduction

Natural Deduction Proof

$P \vdash P \wedge (P \vee Q)$

Proof Script

```
lemma "P  $\implies$  (P  $\wedge$  (P  $\vee$  Q))"  
  apply (rule conjI)  
  (* Subgoal 1 *)  
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  (* Subgoal 2 *)  
  apply (rule disjI1)  
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  done
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Example: Proof by Introduction

Natural Deduction Proof

$$\frac{\text{asm } P \quad \text{disjI1 } P \vdash P \vee Q}{P \vdash P \wedge (P \vee Q)} \text{conjI}$$

Proof Script

```
lemma "P ==> (P ^ (P v Q))"
  apply (rule conjI)
  (* Subgoal 1 *)
  apply assumption
  (* Subgoal 2 *)
  apply (rule disjI1)
  apply assumption
  done
```

Example: Proof by Introduction

Natural Deduction Proof

$$\frac{P \vdash P \quad P \vdash P \vee Q}{P \vdash P \wedge (P \vee Q)} \text{conjI}$$

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Example: Proof by Introduction

Natural Deduction Proof

$$\frac{\frac{}{P \vdash P} \text{asm} \quad \frac{P \vdash P \quad P \vdash P \vee Q}{P \vdash P \vee Q} \text{disjI1}}{P \vdash P \wedge (P \vee Q)} \text{conjI}$$

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lemma "P  $\implies$  (P  $\wedge$  (P  $\vee$  Q))"
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lemma "P  $\implies$  (P  $\wedge$  (P  $\vee$  Q))"
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lemma "P  $\implies$  (P  $\wedge$  (P  $\vee$  Q))"
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  done
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Example: Proof by Introduction

Natural Deduction Proof

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$$\frac{\frac{-}{P \vdash P} \text{asm} \quad \frac{\frac{-}{P \vdash P} \text{asm} \quad P \vdash P \vee Q}{P \vdash P \vee Q} \text{disjI1}}{P \vdash P \wedge (P \vee Q)} \text{conjI}$$

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$$\frac{\frac{-}{P \vdash P} \text{asm} \quad \frac{\frac{-}{P \vdash P} \text{asm} \quad P \vdash P \vee Q}{P \vdash P \vee Q} \text{disjI1}}{P \vdash P \wedge (P \vee Q)} \text{conjI}$$

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lemma "P ==> (P ^ (P v Q))"
  apply (rule conjI)
  (* Subgoal 1 *)
  apply assumption
  (* Subgoal 2 *)
  apply (rule disjI1)
  apply assumption
done
```

Example: Proof by Introduction (Again)

Proof State

```
lemma "P  $\implies$  (P  $\wedge$  (P  $\vee$  Q))"  
  apply (rule conjI)  
    (* Subgoal 1 *)  
  apply assumption  
    (* Subgoal 2 *)  
  apply (rule disjI1)  
  apply assumption  
done
```

Example: Proof by Introduction (Again)

Proof State

```
lemma "P  $\implies$  (P  $\wedge$  (P  $\vee$  Q))"  
  apply (rule conjI)  
    (* Subgoal 1 *)  
  apply assumption  
    (* Subgoal 2 *)  
  apply (rule disjI1)  
  apply assumption  
done
```

Example: Proof by Introduction (Again)

Proof State

1 $P \implies (P \wedge (P \vee Q)).$

lemma "P \implies (P \wedge (P \vee Q))"

```
  apply (rule conjI)
  (* Subgoal 1 *)
  apply assumption
  (* Subgoal 2 *)
  apply (rule disjI1)
  apply assumption
done
```

Example: Proof by Introduction (Again)

Proof State

1 $P \implies P.$

2 $P \implies P \vee Q.$

```
lemma "P  $\implies$  (P  $\wedge$  (P  $\vee$  Q))"  
  apply (rule conjI)  
    (* Subgoal 1 *)  
    apply assumption  
    (* Subgoal 2 *)  
    apply (rule disjI1)  
    apply assumption  
  done
```

Example: Proof by Introduction (Again)

Proof State

1 $P \implies P.$

2 $P \implies P \vee Q.$

```
lemma "P  $\implies$  (P  $\wedge$  (P  $\vee$  Q))"  
  apply (rule conjI)  
  (* Subgoal 1 *)  
  apply assumption  
  (* Subgoal 2 *)  
  apply (rule disjI1)  
  apply assumption  
  done
```

Example: Proof by Introduction (Again)

Proof State

1 $P \Rightarrow P \vee Q.$

```
lemma "P  $\Rightarrow$  (P  $\wedge$  (P  $\vee$  Q))"
```

```
  apply (rule conjI)
```

```
  (* Subgoal 1 *)
```

```
  apply assumption
```

```
  (* Subgoal 2 *)
```

```
  apply (rule disjI1)
```

```
  apply assumption
```

```
done
```

Example: Proof by Introduction (Again)

Proof State

1 $P \Rightarrow P \vee Q.$

```
lemma "P  $\Rightarrow$  (P  $\wedge$  (P  $\vee$  Q))"
```

```
  apply (rule conjI)
```

```
  (* Subgoal 1 *)
```

```
  apply assumption
```

```
  (* Subgoal 2 *)
```

```
  apply (rule disjI1)
```

```
  apply assumption
```

```
done
```

Example: Proof by Introduction (Again)

Proof State

1 $P \Rightarrow P.$

```
lemma "P  $\Rightarrow$  (P  $\wedge$  (P  $\vee$  Q))"  
  apply (rule conjI)  
  (* Subgoal 1 *)  
  apply assumption  
  (* Subgoal 2 *)  
  apply (rule disjI1)  
  apply assumption  
  done
```

Example: Proof by Introduction (Again)

Proof State

No subgoals!

```
lemma "P  $\implies$  (P  $\wedge$  (P  $\vee$  Q))"  
  apply (rule conjI)  
  (* Subgoal 1 *)  
  apply assumption  
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Example: Proof by Introduction (Again)

Proof State

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lemma "P  $\implies$  (P  $\wedge$  (P  $\vee$  Q))"  
  apply (rule conjI)  
    (* Subgoal 1 *)  
  apply assumption  
    (* Subgoal 2 *)  
  apply (rule disjI1)  
  apply assumption  
done
```

More Introduction Rules

Introduction Rules

```
lemma "P  $\longrightarrow$  (P  $\wedge$  (True = True))"  
apply (rule impI)  (* P  $\implies$  (P  $\wedge$  (True = True)) *)  
apply (rule conjI) (* P  $\implies$  P and P  $\implies$  (True = True)) *)  
apply (assumption) (* P  $\implies$  (True = True)) *)  
apply (rule refl)  
done
```

More Introduction Rules

Introduction Rules

$$\frac{P, \Gamma \vdash Q}{\Gamma \vdash P \longrightarrow Q} \text{impI} \quad \frac{\Gamma \vdash P \longrightarrow Q \quad \Gamma \vdash Q \longrightarrow P}{\Gamma \vdash P \longleftrightarrow Q} \text{iffI} \quad \frac{-}{\Gamma \vdash t = t} \text{refl}$$

```
lemma "P  $\longrightarrow$  (P  $\wedge$  (True = True))"  
apply (rule impI) (* P  $\implies$  (P  $\wedge$  (True = True)) *)  
apply (rule conjI) (* P  $\implies$  P and P  $\implies$  (True = True)) *)  
apply (assumption) (* P  $\implies$  (True = True)) *)  
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More Introduction Rules

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```
lemma "P  $\longrightarrow$  (P  $\wedge$  (True = True))"
apply (rule impI) (* P  $\implies$  (P  $\wedge$  (True = True)) *)
apply (rule conjI) (* P  $\implies$  P and P  $\implies$  (True = True) *)
apply (assumption) (* P  $\implies$  (True = True) *)
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apply (assumption) (* P  $\implies$  (True = True)) *)
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More Introduction Rules

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$$\frac{P, \Gamma \vdash Q}{\Gamma \vdash P \longrightarrow Q} \text{impI} \quad \frac{\Gamma \vdash P \longrightarrow Q \quad \Gamma \vdash Q \longrightarrow P}{\Gamma \vdash P \longleftrightarrow Q} \text{iffI} \quad \frac{-}{\Gamma \vdash t = t} \text{refl}$$

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lemma "P  $\longrightarrow$  (P  $\wedge$  (True = True))"
apply (rule impI) (* P  $\implies$  (P  $\wedge$  (True = True)) *)
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apply (assumption) (* P  $\implies$  (True = True)) *)
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Elimination Rules

- What can we deduce from hypothesis P ? Reine's subgoal

Elimination Rules for Propositional Calculus

- Elimination rules use the rule tactical (erule / conID).

- Find first hypothesis matching first premise of elimination rule.

- Resolves subgoal with remaining premises: $[A \vee C] \Rightarrow B$.

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- What can we deduce from hypothesis P_i ? **Refine** a subgoal.

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- Elimination rules use **erule** tactical **apply** (erule conjE).
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Example: Proof by Elimination (1)

How do we prove $P \wedge (P \longrightarrow Q) \longrightarrow Q$?

Natural Deduction Proof

```
1. P ∧ (P → Q)      2. P
1 ∧ E                2 E
3. P                 2 E
4. P → Q             1 ∧ E
5. Q                  3 4 → E
6. P ∧ (P → Q) → Q  5 → I
```

Proof Script

```
lemma "P ∧ (P → Q) → Q"
  apply (rule impI)
  apply (erule conjE)
  apply (erule impE)
    (* Subgoal 1 *)
    apply assumption
    (* Subgoal 2 *)
    apply assumption
  done
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$$\begin{array}{l} \text{asm} \quad \text{---} \\ P \Longrightarrow P \quad \text{asm} \quad \text{---} \\ \text{impE} \quad \llbracket P; Q \rrbracket \Longrightarrow Q \\ \llbracket (P \longrightarrow Q); P \rrbracket \Longrightarrow Q \\ \text{conjE} \quad P \wedge (P \longrightarrow Q) \Longrightarrow Q \\ \text{impI} \quad P \wedge (P \longrightarrow Q) \longrightarrow Q \end{array}$$

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$$\frac{\begin{array}{l} \text{asm } P \Rightarrow P \\ \text{impE } [(P \longrightarrow Q); P] \Rightarrow Q \\ \text{conjE } P \wedge (P \longrightarrow Q) \Rightarrow Q \end{array}}{P \wedge (P \longrightarrow Q) \longrightarrow Q} \text{impI}$$

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$$\frac{P \wedge (P \longrightarrow Q) \Longrightarrow Q}{P \wedge (P \longrightarrow Q) \longrightarrow Q} \text{impI}$$

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Natural Deduction Proof

$$\frac{\frac{\frac{P \Rightarrow P}{\text{asm}} \quad \frac{[P; Q] \Rightarrow Q}{\text{asm}}}{[(P \longrightarrow Q); P] \Rightarrow Q}{\text{conjE}} \quad \frac{P \wedge (P \longrightarrow Q) \Rightarrow Q}{P \wedge (P \longrightarrow Q) \longrightarrow Q} \text{impI}$$

Proof Script

```
lemma "P ∧ (P → Q) → Q"
  apply (rule impI)
  apply (erule conjE)
  apply (erule impE)
  (* Subgoal 1 *)
  apply assumption
  (* Subgoal 2 *)
  apply assumption
done
```

Example: Proof by Elimination (1)

How do we prove $P \wedge (P \longrightarrow Q) \longrightarrow Q$?

Natural Deduction Proof

$$\frac{\frac{\frac{P \implies P}{\text{asm}} \quad \frac{[P; Q] \implies Q}{\text{asm}}}{[(P \longrightarrow Q); P] \implies Q}{\frac{P \wedge (P \longrightarrow Q) \implies Q}{\text{conjE}}} \implies Q \quad \text{impI}$$
$$\frac{P \wedge (P \longrightarrow Q) \implies Q}{P \wedge (P \longrightarrow Q) \longrightarrow Q} \text{impI}$$

Proof Script

```
lemma "P ∧ (P → Q) → Q"
  apply (rule impI)
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Example: Proof by Elimination (1)

How do we prove $P \wedge (P \longrightarrow Q) \longrightarrow Q$?

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$$\frac{\frac{\frac{\text{asm} \overline{P \Rightarrow P} \quad \text{asm} \overline{\llbracket P; Q \rrbracket \Rightarrow Q}}{\llbracket (P \longrightarrow Q); P \rrbracket \Rightarrow Q} \text{impE}}{\frac{P \wedge (P \longrightarrow Q) \Rightarrow Q}{P \wedge (P \longrightarrow Q) \longrightarrow Q} \text{conjE}} \text{impI}$$

Proof Script

```
lemma "P ∧ (P → Q) → Q"
  apply (rule impI)
  apply (erule conjE)
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Example: Proof by Elimination (1)

How do we prove $P \wedge (P \longrightarrow Q) \longrightarrow Q$?

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$$\frac{\frac{\frac{P \implies P \quad \llbracket P; Q \rrbracket \implies Q}{\llbracket (P \longrightarrow Q); P \rrbracket \implies Q} \text{impE}}{P \wedge (P \longrightarrow Q) \implies Q} \text{conjE}}{P \wedge (P \longrightarrow Q) \longrightarrow Q} \text{impI}$$

Proof Script

```
lemma "P ∧ (P ⟶ Q) ⟶ Q"
  apply (rule impI)
  apply (erule conjE)
  apply (erule impE)
    (* Subgoal 1 *)
    apply assumption
  (* Subgoal 2 *)
  apply assumption
done
```

Example: Proof by Elimination (1)

How do we prove $P \wedge (P \longrightarrow Q) \longrightarrow Q$?

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$$\frac{\frac{\frac{P \Rightarrow P}{\text{asm}} \quad \frac{[P; Q] \Rightarrow Q}{\text{asm}}}{\frac{[(P \longrightarrow Q); P] \Rightarrow Q}{\text{conjE}}} \text{impE} \quad \frac{P \wedge (P \longrightarrow Q) \Rightarrow Q}{\text{impI}} \quad \frac{P \wedge (P \longrightarrow Q) \longrightarrow Q}{\text{conjE}}$$

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Example: Proof by Elimination (1)

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$$\frac{\frac{\frac{\overline{P} \text{ asm}}{P \Longrightarrow P} \quad \frac{\frac{\overline{[P; Q]} \text{ asm}}{[P; Q] \Longrightarrow Q} \text{ impE}}{[(P \longrightarrow Q); P] \Longrightarrow Q} \text{ conjE}}{P \wedge (P \longrightarrow Q) \Longrightarrow Q} \text{ impI}}{P \wedge (P \longrightarrow Q) \longrightarrow Q}$$

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How do we prove $P \wedge (P \longrightarrow Q) \longrightarrow Q$?

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$$\frac{\frac{\frac{}{P \Rightarrow P} \text{asm}}{\frac{[(P \longrightarrow Q); P] \Rightarrow Q}{P \wedge (P \longrightarrow Q) \Rightarrow Q} \text{conjE}} \text{impE} \quad \frac{}{[P; Q] \Rightarrow Q} \text{asm}}{P \wedge (P \longrightarrow Q) \longrightarrow Q} \text{impI}$$

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$$\frac{\frac{\frac{}{P \Longrightarrow P} \text{asm}}{\frac{[(P \longrightarrow Q); P] \Longrightarrow Q} \text{conjE}} \text{impI}}{\frac{[P; Q] \Longrightarrow Q} \text{impE}} \text{asm}$$

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done
```

Example: Proof by Elimination (2): $(P \wedge P) \longleftrightarrow P$



Proof Script

```
lemma "(P ∧ P) ↔ P"
  apply (rule iffI)
  apply (erule conjE)
  apply assumption
  apply (rule conjI)
  apply assumption
  apply assumption
done
```

Example: Proof by Elimination (2): $(P \wedge P) \longleftrightarrow P$

$$\begin{array}{c} \text{asm} \quad \overline{\quad} \\ \llbracket P; P \rrbracket \Rightarrow P \quad \text{asm} \quad \overline{\quad} \quad P \Rightarrow P \quad \text{asm} \quad \overline{\quad} \quad P \Rightarrow P \\ \text{conjE} \quad \overline{\quad} \quad P \Rightarrow P \quad \text{conjI} \quad \overline{\quad} \quad P \Rightarrow P \wedge P \\ \text{iffI} \quad \overline{\quad} \quad P \wedge P \Rightarrow P \end{array}$$

$(P \wedge P) \longleftrightarrow P$

Proof Script

```
lemma "(P ∧ P) ⟷ P"
  apply (rule iffI)
  apply (erule conjE)
  apply assumption
  apply (rule conjI)
  apply assumption
  apply assumption
done
```

Example: Proof by Elimination (2): $(P \wedge P) \longleftrightarrow P$

$$\frac{\begin{array}{ccc} \text{asm} \overline{P} & \text{asm} \overline{P} & \text{asm} \overline{P} \\ \text{conjE} \Rightarrow P & \text{conjI} \Rightarrow P & \text{iffI} \\ P \wedge P \Rightarrow P & P \Rightarrow P \wedge P & \end{array}}{(P \wedge P) \longleftrightarrow P} \text{iffI}$$

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lemma "(P ∧ P) ↔ P"
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Example: Proof by Elimination (2): $(P \wedge P) \longleftrightarrow P$

$$\begin{array}{c}
 \frac{\text{asm} \overline{[P; P]} \Rightarrow P}{P \wedge P \Rightarrow P} \text{ conjE} \qquad \frac{\text{asm} \overline{P} \Rightarrow P \quad \text{asm} \overline{P} \Rightarrow P}{P \Rightarrow P \wedge P} \text{ conjI} \\
 \hline
 (P \wedge P) \longleftrightarrow P \quad \text{iffI}
 \end{array}$$

Proof Script

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lemma "(P ∧ P) ⟷ P"
  apply (rule iffI)
  apply (erule conjE)
  apply assumption
  apply (rule conjI)
  apply assumption
  apply assumption
  done
  
```

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  apply (rule iffI)
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  apply assumption
  done
  
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$$\begin{array}{c}
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 \hline
 (P \wedge P) \longleftrightarrow P \quad \text{iffI}
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Proof Script

```

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 \hline
 (P \wedge P) \longleftrightarrow P
 \end{array}$$

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  apply (rule iffI)
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  apply assumption
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$$\frac{
 \frac{
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 \quad
 \frac{
 \frac{
 \frac{}{P \Rightarrow P} \text{asm}
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```

lemma "(P ∧ P) ⟷ P"
  apply (rule iffI)
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  apply assumption
  apply assumption
done
  
```

Outline

- 1 Low-Level Proof Scripts
- 2 Natural Deduction Rules for Propositional Calculus
- 3 Automation with the Classical Reasoner

Natural Deduction in Isar

```
lemma "P  $\wedge$  (P  $\longrightarrow$  Q)  $\longrightarrow$  Q"  
proof (rule impI)  
  assume "P  $\wedge$  (P  $\longrightarrow$  Q)"  
  hence "P  $\longrightarrow$  Q" "P"  
    by (erule_tac conjE, simp_all)  
  thus "Q"  
    by (erule_tac impE, simp_all)  
qed
```

- Benefit of being readable without seeing the proof state.
- A little too verbose for simple predicates.

Natural Deduction in Isar

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lemma "P  $\wedge$  (P  $\longrightarrow$  Q)  $\longrightarrow$  Q"  
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Benefit of being readable without seeing the proof state.

A rule too verbose for simple predicates.

Natural Deduction in Isar

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lemma "P  $\wedge$  (P  $\longrightarrow$  Q)  $\longrightarrow$  Q"  
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  assume "P  $\wedge$  (P  $\longrightarrow$  Q)"  
  hence "P  $\longrightarrow$  Q" "P"  
    by (erule_tac conjE, simp_all)  
  thus "Q"  
    by (erule_tac impE, simp_all)  
qed
```

Benefit of being readable without seeing the proof state

A rule too weak for simple predicates

Natural Deduction in Isar

```
lemma "P  $\wedge$  (P  $\longrightarrow$  Q)  $\longrightarrow$  Q"  
proof (rule impI)  
  assume "P  $\wedge$  (P  $\longrightarrow$  Q)"  
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```

Benefit of being explicit with respect to the goal state
A simple way to formalize natural deduction

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```

qed

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Automating Natural Deduction

- Isn't all this low level deduction too much work?
- Fortunately, Isabelle automates natural deduction using the **blast** tactic.
- **Tableaux prover**: represent proof tree structure.

Example

```
lemma "P  $\longrightarrow$  (P  $\wedge$  (P  $\vee$  Q))" by blast
lemma "P  $\wedge$  (P  $\longrightarrow$  Q)  $\longrightarrow$  Q" by blast
```

- Blast searches **proof tree** using introduction and elimination rules.
- `blast intro: thms elim: thms, or [intro], [elim], and [dest].`
- **Safe** rules marked with "!", e.g. `[intro!]`.
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Automating Natural Deduction

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Combining Deduction and Simplification with Auto

- auto is a powerful tactic combining deduction (eg, `assum`) and simplification (eg, `simplify`)
- `auto` applies safe rules repeatedly, simplifying if possible
- Won't lose information as long as the specified safe rules are truly safe
- General technique for breaking a complex goal into several smaller ones
- $$\text{auto}(\text{goal}) = \text{goal} \longrightarrow x \geq x \wedge x \geq 0$$

 $\text{auto}(\text{simp_rules}, \text{goal}, \text{over_rules}[\text{is_prop}])$
- There are also one-shot versions: `force` and `fastforce`
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- We demonstrated a far more general technique.
- We define bespoke logics in Isabelle and then reason about them.
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- Natural deduction in Isabelle/HOL.
- The classical reasoner.

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