Foundations of Types in Isabelle/HOL

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Overview

1. Relationship between sets and types
2. Defining new types using subsets
3. Description operators
4. Benefits and limitations of types

Outline

1. Relationship between sets and types
2. Defining new types using subsets
3. Description operators
4. Benefits and limitations of types

Types as Sets

- Every type `T` has corresponding non-empty set as its universe

  `UNIV :: T set`

- Any value `x :: T` is an element of the type's universe `x` $\in$ `UNIV`.
- All HOL types have at least one element, hence

  $\exists x.\, x \in$ `UNIV`   `UNIV_witness`

- Types are like maximal sets, the largest set of well-typed members.
- `(UNIV :: ocean set) = {Atlantic, Pacific, Indian, Arctic}`.

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Example: Equality Proofs

$$x \notin fv(\Gamma)\ \frac{x \in A, \Gamma \vdash x \in B}{\Gamma \vdash A \subseteq B}\ \texttt{subsetI} \qquad \frac{\Gamma \vdash A \subseteq B \qquad \Gamma \vdash B \subseteq A}{\Gamma \vdash A = B}\ \texttt{equalityI}$$

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lemma ocean_UNIV: "UNIV = {Atlantic, Pacific, Indian, Arctic}"
proof (rule equalityI; rule subsetI)
  fix x :: ocean
  assume "x ∈ UNIV"
  show "x ∈ {Atlantic, Pacific, Indian, Arctic}"
    by (cases x; simp) (* Automate Case Analysis *)
next
  fix x :: ocean
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$$\frac{x \in A, \Gamma \vdash x \in B}{\Gamma \vdash A \subseteq B}\ \texttt{subsetI} \quad (x \notin fv(\Gamma)) \qquad \frac{\Gamma \vdash A \subseteq B \qquad \Gamma \vdash B \subseteq A}{\Gamma \vdash A = B}\ \texttt{equalityI}$$

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1. Relationship between sets and types
2. Defining new types using subsets
3. Description operators
4. Benefits and limitations of types

Types Definitions

- **type_synonym**, **datatype**, and **record** create types.
- But there is a low-level mechanism.
- New types can be created from a non-empty subset of an existing type.
- Use the command **typedef**:

  typedef NT = "A :: T set"**by** ...

- Requires a proof that the provided set A is non-empty.
- **typedef** is used internally by both **datatype** and **record**.
- Generates conversion functions:

  $\mathrm{Abs\_NT} :: \mathrm{T} \Rightarrow \mathrm{NT}$

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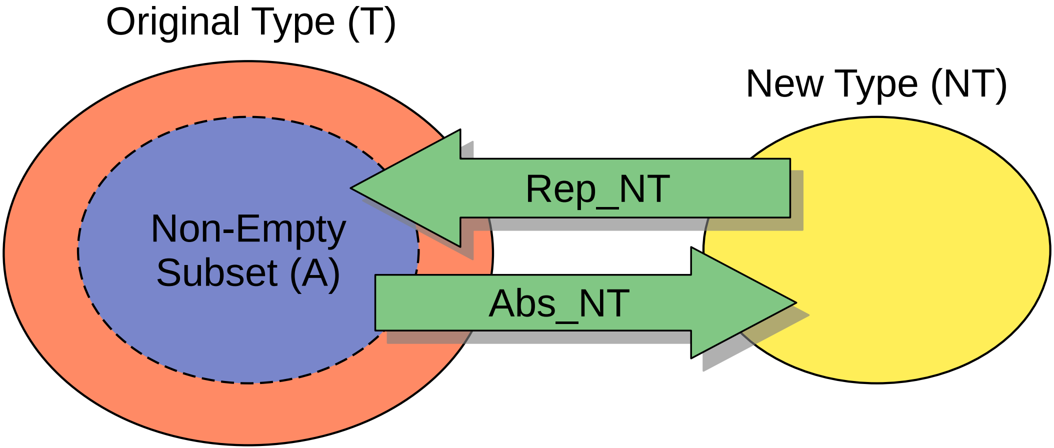

Conversion Functions

- Conversion functions $\texttt{Abs\_NT} :: \texttt{T} \Rightarrow \texttt{NT}$ and $\texttt{Rep\_NT} :: \texttt{NT} \Rightarrow \texttt{T}$.
- They satisfy:

$$\texttt{Abs\_NT}\,(\texttt{Rep\_NT}\ x) = x$$

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Example: Definining Non-Zero Numbers (1)

- Prove that {x :: nat. x > 0} is non-empty by supplying witness 1.
- **typedef** creates from_nat1 :: nat1 ⇒ nat.
- That converts a non-zero nat to a nat.
- And to_nat1 :: nat ⇒ nat1 the does the converse, such that

x > 0 ⟹ from_nat1 (to_nat1 x) = x   (to_nat1_inverse)

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typedef nat1 = "{x :: nat. x > 0}"
  morphisms from_nat1 to_nat1
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Outline

1. Relationship between sets and types
2. Defining new types using subsets
3. Description operators
4. Benefits and limitations of types

Description Operators

- No accident that types are non-empty, but an integral part of the logic.
- Hilbert's choice: pick a value $x :: T$ such that $P\ x$ holds.
- Indefinite article in natural language: "a cat sitting on my roof".
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Example: Indefinite Descriptions

Indefinite Description Equality Theorem (of `someI`)

$$x \notin \mathit{fv}(P,\Gamma)\ \frac{\Gamma \vdash P(a) \qquad P(x),\Gamma \vdash x = a}{\Gamma \vdash (\epsilon\, x.\, P\, x) = a}\ \texttt{some\_equality}$$

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lemma nat_less1_0: "(SOME x :: nat. x < 1) = 0"
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Definite Descriptions

- Gives the unique value x described by P.
- Like definite article in natural language: “the cat sitting on my roof”.
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Definite Descriptions

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Introduction Rule

$$y \notin \mathit{fv}(\Gamma, P, Q)\ \frac{\Gamma \vdash P(t) \qquad P(y), \Gamma \vdash y = t \qquad P(y), \Gamma \vdash Q(y)}{\Gamma \vdash Q(\iota\, x.\, P(x))}\ \texttt{theI2}$$

Outline

1. Relationship between sets and types
2. Defining new types using subsets
3. Description operators
4. Benefits and limitations of types

Types vs. Sets

- Types and sets seem quite similar. Why have both?
  - 'a set vs. $\mathbb{P}\,A$, 'a × 'b vs. $A \times B$ etc.
- Types resolve problems in naive set theory.
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Russell's Paradox (Proof Attempt)

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`not_not`

$\Gamma \vdash \neg(R \notin R)$

`mem_Collect_eq`

$\Gamma \vdash R \notin \{x \mid x \notin x\}$

`R_def`

$\Gamma \vdash R \notin R$

β-reduction

$\Gamma \vdash (\lambda x.\, x \notin x)\, R$

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\lightning & \\
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- But no equivalents for types.
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Next Lecture

- Automation and sledgehammer.

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- Defining new types using subsets.
- Description operators.
- Benefits and limitations of types.

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Conclusion

This Lecture

- Relationship between sets and types.
- Defining new types using subsets.
- Description operators.
- Benefits and limitations of types.

Next Lecture

- Automation and sledghammer.