# Automation and Sledgehammer in Isabelle

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# Overview

1. SAT solvers, resolution provers, and SMT solvers
2. Tool integration with `sledgehammer`
3. Counterexample generators

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# Automated Proof so Far: Simplifier (`simp`)

- Equational rewriting of propositions ($f(x, y) = g(x, y)$).
- Very fast, and can prove many statements.
- Can't generally handle formulae with quantifiers ($\exists\, x\, y.\, 2x + 3y > n$).
- Classical Reasoner (`blast`, `auto`, `force`, etc.)
- Employs the tableau method to find natural deduction proofs.
- Slower than the simplifier, due to backtracking, but more powerful.
- Can prove many statements involving quantifiers with witnesses.
- However, these tools are often not enough, for example:
- Finding of suitable witnesses for variables in quantified formulae.
- Determining the right set of rules to use from HOL's theorem library.
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# SAT Solvers

- Determine satisfiability of proposition with only Boolean variables.
- $P \wedge \neg Q$ – satisfiable, with $P = True$ and $Q = False$ (the assignment).
- $P \wedge \neg P$ – not satisfiable, no value for $P$ that yields $True$.
- A SAT solver typically returns :
  1. SAT + an assignment or 2. UNSAT
- NP-complete problem.
- Very efficient solvers exist handling millions of variables.
- MiniSat, zChaff, PicoSAT, BerkMin, Lingeling, Glucose, SAT4J.
- Widely used in hardware verification and circuit design.
- Instance of constraint satisfaction problems (see module CORG).

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# Resolution Provers

- Resolution: deduction rule + algorithm for proving 1st-order predicates.
- Modern SAT solvers can produce a resolution-based proof of UNSAT.
- Resolution implemented in some 1st order automated theorem provers:
- Prover9, E, Vampire, SPASS, Waldmeister, and the `metis` proof method
- Clause normal form represents knowledge: conjunction of disjunctions.
- Inherently classical in nature, as it depends on proof by contradiction.

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# Resolution

- `metis` converts HOL proposition into required form, applies resolution.
- Requires all background theorems needed to be passed as hypotheses.
- If a refutation of $\neg G$ emerges, this is translated to a HOL theorem.

# Resolution

## Algorithm (Outline)

1. Hypotheses $A_1 \cdots A_n$ and goal $G$: produce $A_1 \wedge \cdots \wedge A_n \wedge \neg G$.
2. Rewrite $A_i$ and $G$ into CNF: $P \wedge (P \longrightarrow Q)$ becomes $P \wedge (\neg P \vee Q)$.
3. Apply the resolution rule over and over to make all possible deductions.
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# Resolution Example

- We want to prove $man(S), \forall x.\ man(x) \longrightarrow mortal(x) \vdash mortal(S)$.
- Rewrite to CNF gives

  $A_1 = man(S),\ A_2 = \neg man(x) \vee mortal(x),\ C = mortal(S)$.
- We need to show $A_1 \wedge A_2 \wedge \neg C$ yields a contradiction.
- Resolving $A_1$ and $A_2$ yields the additional clause $A_3 = mortal(S)$.
- Resolving $A_3$ and $C$ yields the empty set, thus the proof is complete.

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$$A_1 = man(S),\ A_2 = \neg man(x) \vee mortal(x),\ G = mortal(S).$$

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- SMT = Satisfiability Modulo Theories.
- Extends SAT with further types and decision procedures:
- Quantifiers, linear arithmetic, bit vectors, arrays, datatypes, records, etc.
- UNSAT: proof term often returned using accompanying ATP.
- More readily applicable to program verification.
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## Presburger Arithmetic (`presburger`)

- Arithmetic formulae involving only $0$, $1$, and $+$, and equality.
- Decidable with exponential complexity.
- `presburger` handles quantifiers using quantifier elimination (QE).
- E.g. $\neg(\exists\, x\; y : \mathbb{N}.\, 2x + 5y = 3)$ is solvable with `presburger`.

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# Overview

1. SAT solvers, resolution provers, and SMT solvers
2. Tool integration with `sledgehammer`
3. Counterexample generators

# Sledgehammer

- Integration of resolution provers and SMT solvers into Isabelle/HOL.
- Use the “Sledgehammer” pane to apply the tool on an open subgoal.
- Runs several tools in parallel, reconstructs output as Isabelle proof.
- Brings benefits of proof automation to interactive theorem proving.
- Does so in a safe way – there is no need to trust the external tools.

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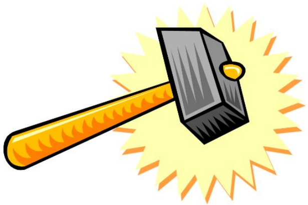

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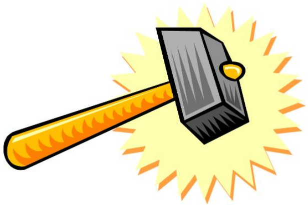

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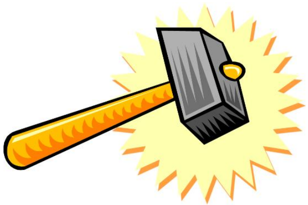

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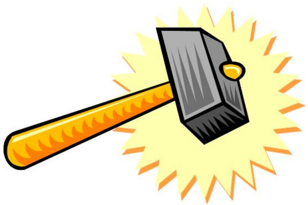

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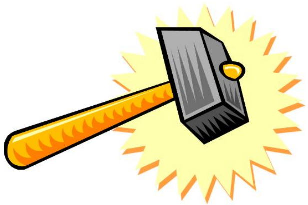

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lemma "(A < B)= (A !≠! B !∧! (!∀! x!∈!A. x !∈! B))"
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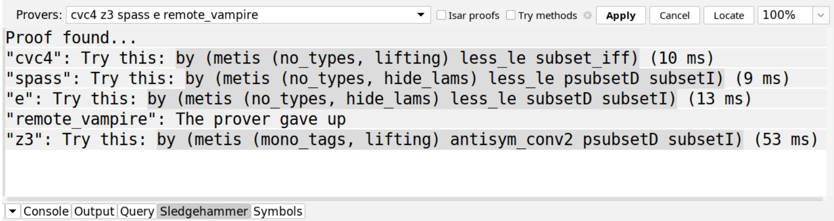


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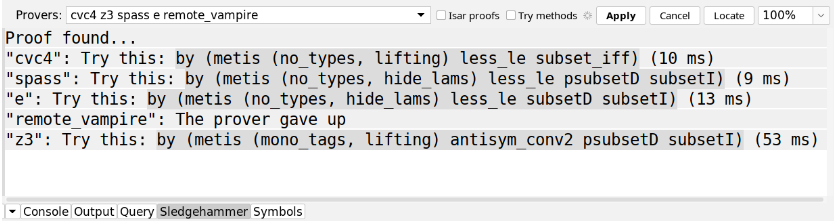


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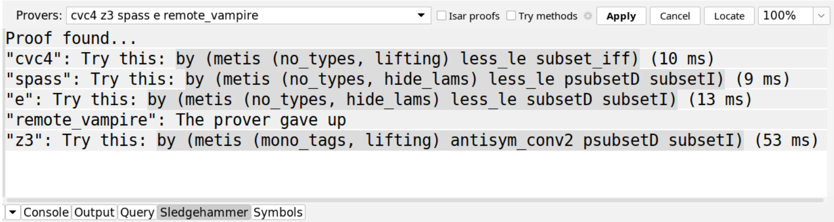


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  by (metis (no_types, lifting) less_le subset_iff)
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```
lemma rat_prop:
  fixes x :: rat
  shows "x² - 3*x + 2 < 0 ⟶ x > 0"
  by (metis add_less_zeroD add_neg_neg
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# Sledgehammer Examples (1)

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lemma tl_element:
  assumes "x ∈ set xs" "x ≠ hd(xs)"
  shows "x ∈ set(tl(xs))"
  by (metis assms(1) assms(2) list.exhaust_sel
     list.sel(2) set_ConsD)
```

# Sledgehammer Examples (2)

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```
lemma sorted_distinct:
  assumes "sorted xs" "distinct xs"
  shows "(∀ i<length xs - 1. xs i < xs(i + 1))"
using assms proof (induct xs)
  case Nil
  then show ?case by simp
next
  case (Cons a xs)
  then show ?case
    by (simp, metis Suc_leI Suc_le_lessD diff_less
      less_nat_zero_code linorder_le_less_linear
      not_one_le_zero nth_Cons' nth_Cons_Suc
      nth_equal_first_eq order_less_le sorted_wrt_nth_less
      strict_sorted_iff)
qed
```

# Overview

1. SAT solvers, resolution provers, and SMT solvers
2. Tool integration with `sledgehammer`
3. Counterexample generators

# Counterexamples

- Automated theorem provers may fail to prove a goal you believe is true
- Could be (1) some lemmas are missing or (2) the goal is actually false.
- A counterexample is a variable assignment that falsifies a theorem:

```
lemma wrong: "(x::nat) > 5"
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- This states that all natural numbers are greater than 5. Not provable.
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Auto Quickcheck found a counterexample:
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  ys = [b]
Evaluated terms:
  rev (xs @ ys) = [b, a]
  rev xs @ rev ys = [a, b]
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# Constraint Solving with Quickcheck

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- Counterexample generator based on a SAT solver.
- Negate the theorem, and pass to the Kodkod relational constraint solver.
- KodKod successfully applied to software verification.
- Example: Hotel Key Card problem.
- Tries to find a finite model that falsifies the theorem.
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