

Natural Deduction Propositional Calculus

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Overview

Natural Deduction Rules

Introduction

Basic Rules

Conjunction

Disjunction

Implication

Equivalence

Negation

Summary

Outline

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$$\frac{-}{P, \Gamma \vdash P} \text{asm} \quad \frac{\Gamma \vdash Q}{P, \Gamma \vdash Q} \text{thin} \quad \frac{\Gamma, P \vdash Q}{P, \Gamma \vdash Q} \text{rotate} \quad \frac{-}{\Gamma \vdash t = t} \text{refl}$$

$$\frac{s = t, \Gamma \vdash P(t)}{s = t, \Gamma \vdash P(s)} \text{subst} \quad \frac{\Gamma \vdash f(x) = g(x) \quad x \notin \text{fv}(\Gamma)}{\Gamma \vdash f = g} \text{ext}$$

$$\frac{-}{\Gamma \vdash \text{true}} \text{TrueI} \quad \frac{-}{\text{false}, \Gamma \vdash P} \text{FalseE}$$

$$\frac{\Gamma \vdash P \quad \Gamma \vdash Q}{\Gamma \vdash P \wedge Q} \text{conjI} \quad \frac{P, Q, \Gamma \vdash R}{P \wedge Q, \Gamma \vdash R} \text{conjE}$$

$$\frac{\Gamma \vdash P}{\Gamma \vdash P \vee Q} \text{disjI1} \quad \frac{\Gamma \vdash Q}{\Gamma \vdash P \vee Q} \text{disjI2} \quad \frac{P, \Gamma \vdash R \quad Q, \Gamma \vdash R}{P \vee Q, \Gamma \vdash R} \text{disjE}$$

$$\frac{P, \Gamma \vdash Q}{\Gamma \vdash P \Rightarrow Q} \text{impI} \quad \frac{\Gamma \vdash P \quad Q, \Gamma \vdash R}{P \Rightarrow Q, \Gamma \vdash R} \text{impE}$$

$$\frac{\Gamma \vdash P \Rightarrow Q \quad \Gamma \vdash Q \Rightarrow P}{\Gamma \vdash P \Leftrightarrow Q} \text{iffI} \quad \frac{P \Rightarrow Q, Q \Rightarrow P, \Gamma \vdash R}{P \Leftrightarrow Q, \Gamma \vdash R} \text{iffE}$$

$$\frac{P, \Gamma \vdash \text{false}}{\Gamma \vdash \neg P} \text{notI} \quad \frac{\Gamma \vdash P}{\neg P, \Gamma \vdash R} \text{notE} \quad \frac{\neg P, \Gamma \vdash \text{false}}{\Gamma \vdash P} \text{ccontr} \quad \frac{\neg Q, \Gamma \vdash P}{\Gamma \vdash P \vee Q} \text{disjCI}$$

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Gentzen Sequent Calculus

- ▶ Style of formal logical argumentation on conditional tautologies: **sequents**.
- ▶ **Natural deduction**: every sequent has exactly one conclusion.
- ▶ $A_1, A_2, \dots, A_n \vdash P$, where the A_i 's and P are propositions and $n \in \mathbb{N}$.
- ▶ Each sequent is **inferred** from sequents on earlier lines in a **formal argument**.
- ▶ Each step appeals to a precise **rule of inference**.
- ▶ **Theorems**: formulas P such that $\vdash P$ is the conclusion of a valid proof.
- ▶ **Convention**: Γ is a list of hypotheses. Example: $\Gamma \vdash P$.

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Inference rules

- ▶ Inferences on sequents use a system of natural deduction rules.
- ▶ A collection of sound inference rules.

Schematic example:

If all of the premises S_i hold, then the goal T must hold:
$$\frac{S_1 \cdots S_n}{T}$$

- ▶ Example:
$$\frac{\Gamma \vdash P \quad \Gamma \vdash Q}{\Gamma \vdash P \wedge Q} \text{ conjI}$$
- ▶ Interpretation: If we have a proof of the validity of $\Gamma \vdash P$ and of $\Gamma \vdash Q$, then we can infer that $\Gamma \vdash P \wedge Q$ is also valid.
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Naming

- ▶ Most of our rules are associated with a particular connective.
- ▶ An **introduction** rule has the connective in the conclusion of its goal.
- ▶ An **elimination** rule has the connective in the hypotheses of its goal.
- ▶ Example: conjunction.

Conjunction introduction:
$$\frac{\Gamma \vdash P \quad \Gamma \vdash Q}{\Gamma \vdash P \wedge Q} \text{ conjI}$$

Conjunction elimination:
$$\frac{P, Q, \Gamma \vdash R}{P \wedge Q, \Gamma \vdash R} \text{ conjE}$$

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- ▶ Shows how a conclusion can be reached from a set of premises.
- ▶ **Formal proof**: every step justified from basic rules.
- ▶ Example:

- ▶ `asm`, `conjI`, `conjE`, `rotate`, and `impI` are all valid rule names properly applied.
- ▶ So it's a formal proof of the theorem $P \wedge Q \Rightarrow Q \wedge P$.
- ▶ Note that complete proof trees have empty premises (leaves).

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- *asm, conjI, conjE, rotate, and impI are all valid rule names properly applied.*
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The Rules of Inference

- ▶ P, Q, R range over propositions.
- ▶ x, y, z range over fixed variables.
- ▶ s, t range over terms.
- ▶ f, g range over total functions.
- ▶ A, B range over sets.
- ▶ Variable Γ is a proof context, consisting of a sequence of propositions, P, Q, \dots, R .
- ▶ Meta-logical function fv returns the set of variables free in a term or context.

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Outline

Natural Deduction Rules

Introduction

Basic Rules

Conjunction

Disjunction

Implication

Equivalence

Negation

Summary

Inference Rule: Assumption

- ▶ The simplest proof rule proves a goal from the hypothesis: $\frac{}{P, \Gamma \vdash P} \text{asm}$
- ▶ The goal $P, \Gamma \vdash P$ is replaced by the empty goal and the proof is finished.

▶ Example

Suppose we want to prove an arbitrary proposition P .

Here's a shallow proof tree that seems to do the job: $\frac{}{P \vdash P} \text{asm}$

Our proof is immediate! But what have we proved?

- ▶ Assuming P is true, we have proved P : that is, $P \vdash P$.
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Manipulating Hypotheses: Thinning

- ▶ Structural Rule.

- ▶ Unnecessary hypotheses can be removed in a proof:

$$\frac{\Gamma \vdash Q}{P, \Gamma \vdash Q} \text{thin}$$

- ▶ Example:

Suppose we know some facts about x : $0 \leq x \leq 10$ and $x \in \mathbb{N}$.

Suppose we need to prove that x is an integer: $x \in \mathbb{Z}$.

We have more hypotheses than we need:

- ▶ The proof will now follow from the fact that $\mathbb{N} \subseteq \mathbb{Z}$.

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- ▶ The proof will now follow from the fact that $\mathbb{N} \subseteq \mathbb{Z}$.

Manipulating Hypotheses: Thinning

- ▶ Structural Rule.

- ▶ Unnecessary hypotheses can be removed in a proof: $\frac{\Gamma \vdash Q}{P, \Gamma \vdash Q} \text{thin}$

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- ▶ One way of completing a branch of a proof is to use **reflexivity**.
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- ▶ This proof illustrates another use of the thinning rule:

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Truth Constants

- ▶ The truth constants are *true* and *false*.
- ▶ They can both be used to complete a proof branch.
- ▶ If our goal has the conclusion *true*, then there is nothing left to prove:

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- ▶ We are trying to prove something from a contradiction, any goal now follows.
- ▶ Compare with implication: $(\textit{false} \Rightarrow P) = (Q \Rightarrow \textit{true}) = \textit{true}$.

Truth Constants

- ▶ The truth constants are *true* and *false*.
- ▶ They can both be used to complete a proof branch.
- ▶ If our goal has the conclusion *true*, then there is nothing left to prove:

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Outline

Natural Deduction Rules

Introduction

Basic Rules

Conjunction

Disjunction

Implication

Equivalence

Negation

Summary

Conjunction

- ▶ As we have already seen, conjunction has two rules of reasoning.

- ▶ **Conjunction introduction:** conjuncts can be proved separately:

$$\frac{\Gamma \vdash P \quad \Gamma \vdash Q}{\Gamma \vdash P \wedge Q} \text{ conjI}$$

- ▶ **Conjunction elimination:** conjuncts in an antecedent can be used separately

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Example: Conjunction is Associative

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Natural Deduction Rules

$$P \wedge (Q \wedge R) \vdash (P \wedge Q) \wedge R$$

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Natural Deduction Rules

$$\frac{}{P \wedge (Q \wedge R) \vdash (P \wedge Q) \wedge R}$$

Example: Conjunction is Associative

Natural Deduction Rules

$$\frac{}{P \wedge (Q \wedge R) \vdash (P \wedge Q) \wedge R} \text{conjE}$$

Example: Conjunction is Associative

$$\frac{P, Q \wedge R \vdash (P \wedge Q) \wedge R}{P \wedge (Q \wedge R) \vdash (P \wedge Q) \wedge R} \text{ conjE}$$

Example: Conjunction is Associative

$$\frac{\overline{P, Q \wedge R \vdash (P \wedge Q) \wedge R}}{P \wedge (Q \wedge R) \vdash (P \wedge Q) \wedge R} \text{ conjE}$$

Example: Conjunction is Associative

$$\frac{\frac{}{P, Q \wedge R \vdash (P \wedge Q) \wedge R} \text{rotate}}{P \wedge (Q \wedge R) \vdash (P \wedge Q) \wedge R} \text{conjE}$$

Example: Conjunction is Associative

$$\frac{\frac{Q \wedge R, P \vdash (P \wedge Q) \wedge R}{P, Q \wedge R \vdash (P \wedge Q) \wedge R} \text{rotate}}{P \wedge (Q \wedge R) \vdash (P \wedge Q) \wedge R} \text{conjE}$$

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Natural Deduction Rules

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$$\frac{\frac{\frac{\frac{Q, R, P \vdash (P \wedge Q) \wedge R}{Q \wedge R, P \vdash (P \wedge Q) \wedge R} \text{conjE}}{P, Q \wedge R \vdash (P \wedge Q) \wedge R} \text{rotate}}{P \wedge (Q \wedge R) \vdash (P \wedge Q) \wedge R} \text{conjE} \text{conjI}$$

Example: Conjunction is Associative

$$\frac{\frac{\frac{Q, R, P \vdash P \wedge Q}{Q, R, P \vdash (P \wedge Q) \wedge R} \text{conjE}}{Q \wedge R, P \vdash (P \wedge Q) \wedge R} \text{rotate}}{\frac{Q, R, P \vdash (P \wedge Q) \wedge R}{P \wedge (Q \wedge R) \vdash (P \wedge Q) \wedge R} \text{conjE}} \text{conjI}$$

Example: Conjunction is Associative

$$\frac{\frac{\frac{Q, R, P \vdash P \wedge Q}{Q, R, P \vdash (P \wedge Q) \wedge R} \text{conjE} \quad \frac{Q, R, P \vdash R}{P, Q \wedge R \vdash (P \wedge Q) \wedge R} \text{rotate}}{P \wedge (Q \wedge R) \vdash (P \wedge Q) \wedge R} \text{conjE} \quad \text{conjI}$$

Example: Conjunction is Associative

Natural Deduction Rules

$$\frac{\frac{\frac{Q, R, P \vdash P \wedge Q}{Q, R, P \vdash (P \wedge Q) \wedge R} \text{conjE} \quad \frac{\frac{Q, R, P \vdash R}{Q, R, P \vdash (P \wedge Q) \wedge R} \text{rotate} \quad \text{conjI}}{P, Q \wedge R \vdash (P \wedge Q) \wedge R} \text{conjE} \quad \text{rotate}}{P \wedge (Q \wedge R) \vdash (P \wedge Q) \wedge R} \text{conjE}$$

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Natural Deduction Rules

$$\frac{\frac{Q, R, P \vdash P \wedge Q}{\frac{Q, R, P \vdash (P \wedge Q) \wedge R}{\frac{Q \wedge R, P \vdash (P \wedge Q) \wedge R}{\frac{P, Q \wedge R \vdash (P \wedge Q) \wedge R}{P \wedge (Q \wedge R) \vdash (P \wedge Q) \wedge R}} \text{conjE}} \text{rotate}} \text{conjE} \quad \frac{\frac{R, P, Q \vdash R}{Q, R, P \vdash R} \text{rotate}}{Q, R, P \vdash R} \text{conjI}$$

Example: Conjunction is Associative

Natural Deduction Rules

$$\frac{\frac{\frac{Q, R, P \vdash P \wedge Q}{Q, R, P \vdash (P \wedge Q) \wedge R} \text{conjE} \quad \frac{\frac{\frac{R, P, Q \vdash R}{Q, R, P \vdash R} \text{rotate}}{Q, R, P \vdash R} \text{asm}}{Q, R, P \vdash R} \text{conjI}}{P, Q \wedge R \vdash (P \wedge Q) \wedge R} \text{rotate} \quad \frac{P \wedge (Q \wedge R) \vdash (P \wedge Q) \wedge R}{P \wedge (Q \wedge R) \vdash (P \wedge Q) \wedge R} \text{conjE}$$

Example: Conjunction is Associative

Natural Deduction Rules

$$\frac{\frac{Q, R, P \vdash P \wedge Q}{\frac{Q, R, P \vdash (P \wedge Q) \wedge R}{\frac{Q \wedge R, P \vdash (P \wedge Q) \wedge R}{P, Q \wedge R \vdash (P \wedge Q) \wedge R} \text{conjE}} \text{conjE}} \text{rotate}} \text{conjI}$$
$$\frac{\frac{\frac{-}{R, P, Q \vdash R} \text{asm}}{Q, R, P \vdash R} \text{rotate}}{Q, R, P \vdash P \wedge Q} \text{conjI}$$

Example: Conjunction is Associative

Natural Deduction Rules

$$\begin{array}{c} \frac{}{Q, R, P \vdash P \wedge Q} \quad \frac{\frac{}{R, P, Q \vdash R} \text{asm}}{Q, R, P \vdash R} \text{rotate} \\ \hline \frac{Q, R, P \vdash (P \wedge Q) \wedge R}{Q \wedge R, P \vdash (P \wedge Q) \wedge R} \text{conjE} \\ \frac{Q \wedge R, P \vdash (P \wedge Q) \wedge R}{P, Q \wedge R \vdash (P \wedge Q) \wedge R} \text{rotate} \\ \hline \frac{P, Q \wedge R \vdash (P \wedge Q) \wedge R}{P \wedge (Q \wedge R) \vdash (P \wedge Q) \wedge R} \text{conjE} \end{array}$$

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$$\begin{array}{c} \frac{}{Q, R, P \vdash P \wedge Q} \text{conjI} \quad \frac{\frac{}{R, P, Q \vdash R} \text{asm}}{Q, R, P \vdash R} \text{rotate} \\ \hline \frac{Q, R, P \vdash P \wedge Q \quad Q, R, P \vdash R}{Q, R, P \vdash (P \wedge Q) \wedge R} \text{conjI} \\ \hline \frac{Q, R, P \vdash (P \wedge Q) \wedge R}{Q \wedge R, P \vdash (P \wedge Q) \wedge R} \text{conjE} \\ \hline \frac{Q \wedge R, P \vdash (P \wedge Q) \wedge R}{P, Q \wedge R \vdash (P \wedge Q) \wedge R} \text{rotate} \\ \hline \frac{P, Q \wedge R \vdash (P \wedge Q) \wedge R}{P \wedge (Q \wedge R) \vdash (P \wedge Q) \wedge R} \text{conjE} \end{array}$$

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$$\begin{array}{c} \frac{Q, R, P \vdash P \quad Q, R, P \vdash Q}{Q, R, P \vdash P \wedge Q} \text{conjI} \quad \frac{\frac{\frac{}{R, P, Q \vdash R} \text{asm}}{Q, R, P \vdash R} \text{rotate}}{Q, R, P \vdash R} \text{conjI} \\ \frac{Q, R, P \vdash (P \wedge Q) \wedge R}{Q \wedge R, P \vdash (P \wedge Q) \wedge R} \text{conjE} \\ \frac{Q \wedge R, P \vdash (P \wedge Q) \wedge R}{P, Q \wedge R \vdash (P \wedge Q) \wedge R} \text{rotate} \\ \frac{P, Q \wedge R \vdash (P \wedge Q) \wedge R}{P \wedge (Q \wedge R) \vdash (P \wedge Q) \wedge R} \text{conjE} \end{array}$$

Example: Conjunction is Associative

Natural Deduction Rules

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Example: Conjunction is Associative

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Example: Conjunction is Associative

$$\begin{array}{c} \frac{}{Q, R, P \vdash P} \quad \frac{}{Q, R, P \vdash Q} \text{asm} \quad \frac{}{R, P, Q \vdash R} \text{asm} \\ \hline \frac{}{Q, R, P \vdash P \wedge Q} \text{conjI} \quad \frac{}{Q, R, P \vdash R} \text{rotate} \\ \hline \frac{}{Q, R, P \vdash (P \wedge Q) \wedge R} \text{conjI} \\ \hline \frac{}{Q \wedge R, P \vdash (P \wedge Q) \wedge R} \text{conjE} \\ \hline \frac{}{P, Q \wedge R \vdash (P \wedge Q) \wedge R} \text{rotate} \\ \hline \frac{}{P \wedge (Q \wedge R) \vdash (P \wedge Q) \wedge R} \text{conjE} \end{array}$$

Example: Conjunction is Associative

$$\begin{array}{c} \frac{}{Q, R, P \vdash P} \text{rotate} \quad \frac{}{Q, R, P \vdash Q} \text{asm} \quad \frac{}{R, P, Q \vdash R} \text{asm} \\ \hline \frac{}{Q, R, P \vdash P \wedge Q} \text{conjI} \quad \frac{}{Q, R, P \vdash R} \text{rotate} \\ \hline \frac{}{Q, R, P \vdash (P \wedge Q) \wedge R} \text{conjI} \\ \hline \frac{}{Q \wedge R, P \vdash (P \wedge Q) \wedge R} \text{conjE} \\ \hline \frac{}{P, Q \wedge R \vdash (P \wedge Q) \wedge R} \text{rotate} \\ \hline \frac{}{P \wedge (Q \wedge R) \vdash (P \wedge Q) \wedge R} \text{conjE} \end{array}$$

Example: Conjunction is Associative

$$\begin{array}{c}
 \frac{R, P, Q \vdash P}{Q, R, P \vdash P} \text{ rotate} \quad \frac{}{Q, R, P \vdash Q} \text{ asm} \quad \frac{}{R, P, Q \vdash R} \text{ asm} \\
 \hline
 \frac{}{Q, R, P \vdash P \wedge Q} \text{ conjI} \quad \frac{}{Q, R, P \vdash R} \text{ rotate} \\
 \hline
 \frac{}{Q, R, P \vdash (P \wedge Q) \wedge R} \text{ conjI} \\
 \hline
 \frac{}{Q \wedge R, P \vdash (P \wedge Q) \wedge R} \text{ conjE} \\
 \hline
 \frac{}{P, Q \wedge R \vdash (P \wedge Q) \wedge R} \text{ rotate} \\
 \hline
 \frac{}{P \wedge (Q \wedge R) \vdash (P \wedge Q) \wedge R} \text{ conjE}
 \end{array}$$

Example: Conjunction is Associative

$$\begin{array}{c} \frac{\frac{R, P, Q \vdash P}{Q, R, P \vdash P} \text{ rotate}}{Q, R, P \vdash P \wedge Q} \text{ conjI} \quad \frac{\frac{}{Q, R, P \vdash Q} \text{ asm}}{Q, R, P \vdash P \wedge Q} \text{ conjI} \quad \frac{\frac{}{R, P, Q \vdash R} \text{ asm}}{Q, R, P \vdash R} \text{ rotate} \\ \frac{Q, R, P \vdash P \wedge Q \quad Q, R, P \vdash R}{Q, R, P \vdash (P \wedge Q) \wedge R} \text{ conjE} \\ \frac{Q, R, P \vdash (P \wedge Q) \wedge R}{Q \wedge R, P \vdash (P \wedge Q) \wedge R} \text{ rotate} \\ \frac{P, Q \wedge R \vdash (P \wedge Q) \wedge R}{P \wedge (Q \wedge R) \vdash (P \wedge Q) \wedge R} \text{ conjE} \end{array}$$

Example: Conjunction is Associative

$$\begin{array}{c}
 \frac{}{R, P, Q \vdash P} \text{rotate} \\
 \frac{}{Q, R, P \vdash P} \text{rotate} \\
 \hline
 Q, R, P \vdash P \wedge Q
 \end{array}
 \quad
 \begin{array}{c}
 \frac{}{Q, R, P \vdash Q} \text{asm} \\
 \hline
 \text{conjI}
 \end{array}
 \quad
 \begin{array}{c}
 \frac{}{R, P, Q \vdash R} \text{asm} \\
 \frac{}{Q, R, P \vdash R} \text{rotate} \\
 \hline
 \text{conjI}
 \end{array}$$

$$\begin{array}{c}
 Q, R, P \vdash (P \wedge Q) \wedge R \\
 \hline
 Q \wedge R, P \vdash (P \wedge Q) \wedge R \\
 \hline
 P, Q \wedge R \vdash (P \wedge Q) \wedge R \\
 \hline
 P \wedge (Q \wedge R) \vdash (P \wedge Q) \wedge R
 \end{array}$$

conjE
 rotate
 conjE

Example: Conjunction is Associative

$$\begin{array}{c} \frac{P, Q, R \vdash P}{R, P, Q \vdash P} \text{rotate} \\ \frac{R, P, Q \vdash P}{Q, R, P \vdash P} \text{rotate} \\ \hline Q, R, P \vdash P \wedge Q \end{array} \quad \begin{array}{c} \frac{}{Q, R, P \vdash Q} \text{asm} \\ \hline Q, R, P \vdash P \wedge Q \end{array} \quad \begin{array}{c} \frac{}{R, P, Q \vdash R} \text{asm} \\ \hline Q, R, P \vdash R \end{array} \quad \begin{array}{c} \frac{}{Q, R, P \vdash R} \text{rotate} \\ \hline Q, R, P \vdash R \end{array} \quad \begin{array}{c} \frac{}{Q, R, P \vdash R} \text{conjI} \\ \hline Q, R, P \vdash R \end{array}$$
$$\begin{array}{c} \frac{Q, R, P \vdash (P \wedge Q) \wedge R}{Q \wedge R, P \vdash (P \wedge Q) \wedge R} \text{conjE} \\ \frac{Q \wedge R, P \vdash (P \wedge Q) \wedge R}{P, Q \wedge R \vdash (P \wedge Q) \wedge R} \text{rotate} \\ \frac{P, Q \wedge R \vdash (P \wedge Q) \wedge R}{P \wedge (Q \wedge R) \vdash (P \wedge Q) \wedge R} \text{conjE} \end{array}$$

Example: Conjunction is Associative

$$\begin{array}{c} \frac{}{P, Q, R \vdash P} \\ \hline \frac{}{R, P, Q \vdash P} \text{ rotate} \\ \hline \frac{}{Q, R, P \vdash P} \text{ rotate} \\ \hline \frac{}{Q, R, P \vdash P \wedge Q} \text{ conjI} \end{array} \quad \frac{}{Q, R, P \vdash Q} \text{ conjI} \quad \frac{}{R, P, Q \vdash R} \text{ conjI}$$
$$\begin{array}{c} \frac{}{Q, R, P \vdash (P \wedge Q) \wedge R} \text{ conjE} \\ \hline \frac{}{Q \wedge R, P \vdash (P \wedge Q) \wedge R} \text{ rotate} \\ \hline \frac{}{P, Q \wedge R \vdash (P \wedge Q) \wedge R} \text{ conjE} \\ \hline \frac{}{P \wedge (Q \wedge R) \vdash (P \wedge Q) \wedge R} \text{ conjE} \end{array}$$

Example: Conjunction is Associative

$$\begin{array}{c} \frac{}{P, Q, R \vdash P} \text{asm} \\ \frac{}{R, P, Q \vdash P} \text{rotate} \\ \frac{}{Q, R, P \vdash P} \text{rotate} \\ \hline Q, R, P \vdash P \wedge Q \end{array} \quad \begin{array}{c} \frac{}{Q, R, P \vdash Q} \text{asm} \\ \hline Q, R, P \vdash P \wedge Q \end{array} \text{conjI} \quad \begin{array}{c} \frac{}{R, P, Q \vdash R} \text{asm} \\ \frac{}{Q, R, P \vdash R} \text{rotate} \\ \hline Q, R, P \vdash P \wedge Q \end{array} \text{conjI}$$
$$\begin{array}{c} \frac{}{Q, R, P \vdash (P \wedge Q) \wedge R} \text{conjE} \\ \frac{}{Q \wedge R, P \vdash (P \wedge Q) \wedge R} \text{rotate} \\ \frac{}{P, Q \wedge R \vdash (P \wedge Q) \wedge R} \text{conjE} \\ \hline P \wedge (Q \wedge R) \vdash (P \wedge Q) \wedge R \end{array}$$

Example: Conjunction is Associative

$$\begin{array}{c} \frac{}{P, Q, R \vdash P} \text{asm} \\ \frac{}{R, P, Q \vdash P} \text{rotate} \\ \frac{}{Q, R, P \vdash P} \text{rotate} \\ \hline Q, R, P \vdash P \wedge Q \end{array} \quad \frac{}{Q, R, P \vdash Q} \text{asm} \quad \frac{}{R, P, Q \vdash R} \text{asm}$$
$$\begin{array}{c} \hline Q, R, P \vdash P \wedge Q \\ \hline Q, R, P \vdash (P \wedge Q) \wedge R \\ \hline Q \wedge R, P \vdash (P \wedge Q) \wedge R \\ \hline P, Q \wedge R \vdash (P \wedge Q) \wedge R \\ \hline P \wedge (Q \wedge R) \vdash (P \wedge Q) \wedge R \end{array}$$

conjI

conjE

rotate

conjE

rotate

conjI

Example: Conjunction is Associative: Longer Proof

$$\frac{\frac{\frac{}{P, Q \wedge R \vdash P} \text{asm}}{P \wedge (Q \wedge R) \vdash P} \text{conjE}}{P \wedge (Q \wedge R) \vdash P \wedge Q} \text{conjI}$$
$$\frac{\frac{\frac{\frac{}{Q, R, P \vdash Q} \text{asm}}{Q \wedge R, P \vdash Q} \text{conjE}}{P, Q \wedge R \vdash Q} \text{rotate}}{P \wedge (Q \wedge R) \vdash Q} \text{conjE}$$
$$\frac{\frac{\frac{\frac{}{R, P, Q \vdash R} \text{asm}}{Q, R, P \vdash R} \text{rotate}}{Q \wedge R, P \vdash R} \text{conjE}}{P, Q \wedge R \vdash R} \text{rotate}$$
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$$\begin{array}{c}
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 \hline
 P \wedge (Q \wedge R) \vdash (P \wedge Q) \wedge R
 \end{array}
 \quad
 \begin{array}{c}
 \frac{\frac{\frac{}{Q, R, P \vdash Q} \text{asm}}{Q \wedge R, P \vdash Q} \text{conjE}}{P, Q \wedge R \vdash Q} \text{rotate} \\
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 \end{array}
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Outline

Natural Deduction Rules

Introduction

Basic Rules

Conjunction

Disjunction

Implication

Equivalence

Negation

Summary

Disjunction

► Disjunction introduction

To prove $P \vee Q$ we need to prove either P or Q :

$$\frac{\Gamma \vdash P}{\Gamma \vdash P \vee Q} \text{disjI1}$$

$$\frac{\Gamma \vdash Q}{\Gamma \vdash P \vee Q} \text{disjI2}$$

► Disjunction elimination

If one of our hypotheses is a disjunction, then we proceed by case analysis:

$$\frac{P, \Gamma \vdash R \quad Q, \Gamma \vdash R}{P \vee Q, \Gamma \vdash R} \text{disjE}$$

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Example: Case Analysis

- Suppose we have an assumption that $(x = a) \vee (x = b)$ in our proof of $P(x)$.
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Example: Disjunction is Commutative

► Proof:

► Oh no!

► The left-hand branch is stuck: in general, we can't prove $P \vdash Q$.

► What went wrong? We threw away information too early in the proof.

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$$\frac{P \vee Q \vdash Q}{P \vee Q \vdash Q \vee P} \text{ disjI1}$$

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- ▶ Let's try again.
- ▶ Proof:
- ▶ Disjunction introduction is called an **unsafe** rule.
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► Proof:

$$\frac{\frac{P \vdash Q \vee P}{P \vee Q \vdash Q \vee P} \text{ disjI2} \quad \frac{\frac{\frac{}{Q \vdash Q} \text{ asm}}{Q \vdash Q \vee P} \text{ disjI1}}{P \vee Q \vdash Q \vee P} \text{ disjE}}$$

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$$\frac{\frac{\frac{}{P \vdash P} \text{asm}}{P \vdash Q \vee P} \text{disjI2} \quad \frac{\frac{\frac{}{Q \vdash Q} \text{asm}}{Q \vdash Q \vee P} \text{disjI1}}{P \vee Q \vdash Q \vee P} \text{disjE}$$

- ▶ Disjunction introduction is called an **unsafe** rule.
- ▶ Heuristic: Apply safe rules before unsafe ones.

Example: Disjunction is Commutative

- ▶ Let's try again.

- ▶ Proof:

$$\frac{\frac{\frac{}{P \vdash P} \text{asm}}{P \vdash Q \vee P} \text{disjI2} \quad \frac{\frac{\frac{}{Q \vdash Q} \text{asm}}{Q \vdash Q \vee P} \text{disjI1}}{P \vee Q \vdash Q \vee P} \text{disjE}$$

- ▶ Disjunction introduction is called an **unsafe** rule.
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Theorems as Inference Rules: the Cut

- ▶ Structural Rule: the cut.
- ▶ Example:
- ▶ We want to prove $\vdash Q \vee P$.
- ▶ Suppose we already have a proof of $\vdash P \vee Q$.
- ▶ We're almost there. If only disjunction was commutative!
- ▶ But we've proved that in the last example: $P \vee Q \vdash Q \vee P$.
- ▶ How do we organise this proof? *We use the cut!*

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► Proof

- Our previous example proved the commutativity lemma $P \vee Q \vdash Q \vee P$.
- The cut justifies using this lemma as an inference rule:
- We can now use this rule in other proofs.
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Outline

Natural Deduction Rules

Introduction

Basic Rules

Conjunction

Disjunction

Implication

Equivalence

Negation

Summary

Implication

- ▶ To prove an implication $P \Rightarrow Q$, we assume P and then prove Q .
- ▶ Implication introduction:
$$\frac{P, \Gamma \vdash Q}{\Gamma \vdash P \Rightarrow Q} \text{ impI}$$
- ▶ What if we have $P \Rightarrow Q$ as a hypothesis?
- ▶ Then we can replace it by Q . But we need to prove P in order to do this.
- ▶ Implication elimination:
$$\frac{\Gamma \vdash P \quad Q, \Gamma \vdash R}{P \Rightarrow Q, \Gamma \vdash R} \text{ impE}$$
- ▶ Implication elimination was called **modus ponens** in antiquity.

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- ▶ The proposition

$$(P \wedge Q \Rightarrow R) \Rightarrow (P \Rightarrow (Q \Rightarrow R))$$

is a theorem of our natural deduction system.

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is a theorem of our natural deduction system.

- Proof:

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$\frac{\frac{}{R, Q, P \vdash R} \text{ asm}}{\text{impE}}$

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$\frac{-}{R, Q, P \vdash R} \text{ asm}$
 impE

Natural Deduction Rules

- $$(P \wedge Q \Rightarrow R) \Rightarrow (P \Rightarrow (Q \Rightarrow R))$$

► Proof:

$$\begin{array}{c}
\frac{}{Q, P \vdash P \wedge Q} \text{conjI} \quad \frac{}{R, Q, P \vdash R} \text{asm} \\
\hline
\frac{P \wedge Q \Rightarrow R, Q, P \vdash R}{P, P \wedge Q \Rightarrow R, Q \vdash R} \text{rotate} \\
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\frac{}{P, P \wedge Q \Rightarrow R, Q \vdash R} \text{rotate} \\
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\hline
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Example: Implication is Transitive

- ▶ We want to prove: $(P \Rightarrow Q) \wedge (Q \Rightarrow R) \Rightarrow (P \Rightarrow R)$.
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Example: Implication is Transitive

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$\frac{\frac{\frac{Q \Rightarrow R, P, Q \vdash R}{Q, Q \Rightarrow R, P \vdash R} \text{rotate}}{\vdash (P \Rightarrow Q) \wedge (Q \Rightarrow R) \Rightarrow (P \Rightarrow R)} \text{impE}$

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\frac{P \Rightarrow Q, Q \Rightarrow R, P \vdash R}{(P \Rightarrow Q) \wedge (Q \Rightarrow R), P \vdash R} \text{ conjE} \\
\frac{(P \Rightarrow Q) \wedge (Q \Rightarrow R), P \vdash R}{P, (P \Rightarrow Q) \wedge (Q \Rightarrow R) \vdash R} \text{ rotate} \\
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Natural Deduction Rules

- $$\begin{array}{c}
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\frac{(P \Rightarrow Q) \wedge (Q \Rightarrow R) \vdash P \Rightarrow R}{\vdash (P \Rightarrow Q) \wedge (Q \Rightarrow R) \Rightarrow (P \Rightarrow R)} \text{impI}
\end{array}$$

Outline

Natural Deduction Rules

Introduction

Basic Rules

Conjunction

Disjunction

Implication

Equivalence

Negation

Summary

Equivalence

- ▶ The equivalence $P \Leftrightarrow Q$ is the bi-implication $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$.
- ▶ This explains the two proof rules, which are related to those for conjunction:

$$\frac{\Gamma \vdash P \Rightarrow Q \quad \Gamma \vdash Q \Rightarrow P}{\Gamma \vdash P \Leftrightarrow Q} \text{ iffI} \qquad \frac{P \Rightarrow Q, Q \Rightarrow P, \Gamma \vdash R}{P \Leftrightarrow Q, \Gamma \vdash R} \text{ iffE}$$

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Example: Rule of Subsumption

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$$\frac{P \Rightarrow Q \vdash P \wedge Q \Rightarrow P \quad \overline{P \Rightarrow Q \vdash P \Rightarrow P \wedge Q}}{P \Rightarrow Q \vdash P \wedge Q \Leftrightarrow P} \text{ iffI}$$

Example: Rule of Subsumption

$$\frac{P \Rightarrow Q \vdash P \wedge Q \Rightarrow P \quad \frac{}{P \Rightarrow Q \vdash P \Rightarrow P \wedge Q} \text{impI}}{P \Rightarrow Q \vdash P \wedge Q \Leftrightarrow P} \text{iffI}$$

Example: Rule of Subsumption

$$\frac{P \Rightarrow Q \vdash P \wedge Q \Rightarrow P \quad \frac{P, P \Rightarrow Q \vdash P \wedge Q}{P \Rightarrow Q \vdash P \Rightarrow P \wedge Q} \text{impI}}{P \Rightarrow Q \vdash P \wedge Q \Leftrightarrow P} \text{iffI}$$

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Example: Rule of Subsumption

$$\frac{\begin{array}{c} P \Rightarrow Q \vdash P \wedge Q \Rightarrow P \\ \hline P \Rightarrow Q \vdash P \wedge Q \Leftrightarrow P \end{array}}{\begin{array}{c} \frac{P \vdash P \quad \frac{Q, P \vdash P \wedge Q}{P \Rightarrow Q, P \vdash P \wedge Q} \text{impE}}{P, P \Rightarrow Q \vdash P \wedge Q} \text{rotate} \\ \frac{P, P \Rightarrow Q \vdash P \wedge Q}{P \Rightarrow Q \vdash P \Rightarrow P \wedge Q} \text{impI} \\ \hline P \Rightarrow Q \vdash P \Rightarrow P \wedge Q \text{iffI} \end{array}}$$

Example: Rule of Subsumption

$$\frac{\frac{\frac{P \vdash P}{P \Rightarrow Q, P \vdash P \wedge Q} \text{ conjI} \quad \frac{P \Rightarrow Q, P \vdash P \wedge Q}{P, P \Rightarrow Q \vdash P \wedge Q} \text{ impE}}{\frac{P, P \Rightarrow Q \vdash P \wedge Q}{P \Rightarrow Q \vdash P \Rightarrow P \wedge Q} \text{ rotate}} \text{ impI}$$
$$\frac{P \Rightarrow Q \vdash P \wedge Q \Rightarrow P \quad P \Rightarrow Q \vdash P \Rightarrow P \wedge Q}{P \Rightarrow Q \vdash P \wedge Q \Leftrightarrow P} \text{ iffI}$$

Example: Rule of Subsumption

$$\frac{\begin{array}{c} P \Rightarrow Q \vdash P \wedge Q \Rightarrow P \\ \hline P \Rightarrow Q \vdash P \wedge Q \Leftrightarrow P \end{array}}{\begin{array}{c} \frac{\frac{P \vdash P}{\frac{P \Rightarrow Q, P \vdash P \wedge Q}{P, P \Rightarrow Q \vdash P \wedge Q} \text{rotate}}{\frac{P \Rightarrow Q \vdash P \Rightarrow P \wedge Q}{P \Rightarrow Q \vdash P \Rightarrow P \wedge Q} \text{iffI}} \quad \frac{\frac{\frac{Q, P \vdash P \quad Q, P \vdash Q}{Q, P \vdash P \wedge Q} \text{conjI}}{P \Rightarrow Q, P \vdash P \wedge Q} \text{impE}}{\frac{P, P \Rightarrow Q \vdash P \wedge Q}{P \Rightarrow Q \vdash P \Rightarrow P \wedge Q} \text{impI}} \end{array}}$$

Example: Rule of Subsumption

$$\frac{\frac{P \Rightarrow Q \vdash P \wedge Q \Rightarrow P}{P \Rightarrow Q \vdash P \wedge Q \Leftrightarrow P} \text{iffI} \quad \frac{\frac{\frac{P \Rightarrow Q, P \vdash P \wedge Q}{P, P \Rightarrow Q \vdash P \wedge Q} \text{rotate} \quad \frac{\frac{P \vdash P \quad \frac{\frac{Q, P \vdash P \quad Q, P \vdash Q}{Q, P \vdash P \wedge Q} \text{conjI}}{Q, P \vdash P \wedge Q} \text{impE}}{P \Rightarrow Q, P \vdash P \wedge Q} \text{impI}}{P \Rightarrow Q \vdash P \Rightarrow P \wedge Q} \text{iffI}}{P \Rightarrow Q \vdash P \wedge Q \Leftrightarrow P} \text{iffI}$$

Example: Rule of Subsumption

$$\begin{array}{c}
 \frac{P \Rightarrow Q \vdash P \wedge Q \Rightarrow P \quad \frac{\frac{P \vdash P \quad \frac{\frac{Q, P \vdash P \quad \overline{Q, P \vdash Q}}{\text{conjI}}}{Q, P \vdash P \wedge Q}}{\text{impE}}}{P \Rightarrow Q, P \vdash P \wedge Q} \text{rotate}}{P \Rightarrow Q \vdash P \Rightarrow P \wedge Q} \text{impI} \\
 \hline
 P \Rightarrow Q \vdash P \wedge Q \Leftrightarrow P \quad \text{iffI}
 \end{array}$$

Example: Rule of Subsumption

$$\begin{array}{c}
 \frac{P \Rightarrow Q \vdash P \wedge Q \Rightarrow P}{P \Rightarrow Q \vdash P \wedge Q \Leftrightarrow P} \\
 \frac{P \Rightarrow Q \vdash P \wedge Q \Rightarrow P \quad \frac{\frac{P, P \Rightarrow Q \vdash P \wedge Q}{P \Rightarrow Q \vdash P \Rightarrow P \wedge Q} \text{impI} \quad \frac{P, P \Rightarrow Q \vdash P \wedge Q}{P, P \Rightarrow Q \vdash P \wedge Q} \text{rotate}}{P \Rightarrow Q \vdash P \Rightarrow P \wedge Q} \text{iffI} \\
 \frac{P \vdash P \quad \frac{\frac{Q, P \vdash P \quad \frac{Q, P \vdash Q}{\text{asm}}}{Q, P \vdash P \wedge Q} \text{conjI}}{P \Rightarrow Q, P \vdash P \wedge Q} \text{impE}}{P \Rightarrow Q, P \vdash P \wedge Q}
 \end{array}$$

Example: Rule of Subsumption

$$\begin{array}{c}
 \frac{P \Rightarrow Q \vdash P \wedge Q \Rightarrow P}{P \Rightarrow Q \vdash P \wedge Q \Leftrightarrow P} \\
 \frac{P \Rightarrow Q \vdash P \wedge Q \Rightarrow P \quad \frac{\frac{P \Rightarrow Q, P \vdash P \wedge Q}{P, P \Rightarrow Q \vdash P \wedge Q} \text{rotate} \quad \frac{P \Rightarrow Q \vdash P \Rightarrow P \wedge Q}{P \Rightarrow Q \vdash P \Rightarrow P \wedge Q} \text{implI}}{P \Rightarrow Q \vdash P \Rightarrow P \wedge Q} \text{iffI} \\
 \frac{P \vdash P \quad \frac{\frac{Q, P \vdash P}{Q, P \vdash P \wedge Q} \text{conjI} \quad \frac{}{Q, P \vdash Q} \text{asm}}{Q, P \vdash P \wedge Q} \text{impE}}{P \Rightarrow Q, P \vdash P \wedge Q} \text{rotate}
 \end{array}$$

Example: Rule of Subsumption

$$\begin{array}{c}
 \frac{P \Rightarrow Q \vdash P \wedge Q \Rightarrow P}{P \Rightarrow Q \vdash P \wedge Q \Leftrightarrow P} \\
 \frac{
 \frac{
 \frac{
 \frac{
 \frac{P \vdash P}{Q, P \vdash P} \text{ rotate}
 }{Q, P \vdash P \wedge Q} \text{ conjI}
 }{P \Rightarrow Q, P \vdash P \wedge Q} \text{ impE}
 }{P, P \Rightarrow Q \vdash P \wedge Q} \text{ rotate}
 }{P \Rightarrow Q \vdash P \Rightarrow P \wedge Q} \text{ impI}
 }{P \Rightarrow Q \vdash P \Rightarrow P \wedge Q} \text{ iffI}
 }{P \Rightarrow Q \vdash P \wedge Q \Rightarrow P}
 \end{array}$$

Example: Rule of Subsumption

$$\begin{array}{c}
 \frac{P \Rightarrow Q \vdash P \wedge Q \Rightarrow P}{P \Rightarrow Q \vdash P \wedge Q \Leftrightarrow P} \\
 \frac{
 \frac{
 \frac{
 \frac{
 \frac{P \vdash P}{P \Rightarrow Q, P \vdash P \wedge Q} \text{ rotate}
 }{P, P \Rightarrow Q \vdash P \wedge Q} \text{ rotate}
 }{P \Rightarrow Q \vdash P \Rightarrow P \wedge Q} \text{ impI}
 }{P \Rightarrow Q \vdash P \wedge Q \Rightarrow P} \text{ iffI}
 }{
 \frac{
 \frac{
 \frac{
 \frac{P \vdash P}{Q, P \vdash P \wedge Q} \text{ impE}
 }{Q, P \vdash P} \text{ conjI}
 }{Q, P \vdash P} \text{ rotate}
 }{Q, P \vdash P} \text{ asm}
 }{Q, P \vdash P}
 }
 \end{array}$$

Example: Rule of Subsumption

$$\begin{array}{c}
 \frac{P \Rightarrow Q \vdash P \wedge Q \Rightarrow P}{P \Rightarrow Q \vdash P \wedge Q \Leftrightarrow P} \\
 \frac{P \vdash P \quad \frac{\frac{P, Q \vdash P}{Q, P \vdash P} \text{ rotate} \quad \frac{-}{Q, P \vdash Q} \text{ asm}}{Q, P \vdash P \wedge Q} \text{ conjI}}{P \Rightarrow Q, P \vdash P \wedge Q} \text{ impE} \\
 \frac{\frac{P, P \Rightarrow Q \vdash P \wedge Q}{P \Rightarrow Q \vdash P \Rightarrow P \wedge Q} \text{ rotate}}{P \Rightarrow Q \vdash P \Rightarrow P \wedge Q} \text{ impI} \\
 \text{iffI}
 \end{array}$$

Example: Rule of Subsumption

$$\begin{array}{c}
 \frac{P \Rightarrow Q \vdash P \wedge Q \Rightarrow P}{P \Rightarrow Q \vdash P \wedge Q \Leftrightarrow P} \\
 \frac{P \vdash P \quad \frac{\frac{P \Rightarrow Q, P \vdash P \wedge Q}{P, P \Rightarrow Q \vdash P \wedge Q} \text{rotate} \quad \frac{P \Rightarrow Q \vdash P \Rightarrow P \wedge Q}{P \Rightarrow Q \vdash P \Rightarrow P \wedge Q} \text{impI}}{P \Rightarrow Q, P \vdash P \wedge Q} \text{impE} \quad \frac{\frac{\frac{P, Q \vdash P}{Q, P \vdash P} \text{rotate} \quad \frac{P, Q \vdash P \wedge Q}{Q, P \vdash P \wedge Q} \text{conjI}}{Q, P \vdash Q} \text{asm} \quad \frac{}{Q, P \vdash Q} \text{asm}}{P \Rightarrow Q, P \vdash P \wedge Q} \text{impE}
 \end{array}$$

Example: Rule of Subsumption

$$\begin{array}{c}
 \frac{P \Rightarrow Q \vdash P \wedge Q \Rightarrow P}{P \Rightarrow Q \vdash P \wedge Q \Leftrightarrow P} \\
 \frac{P \Rightarrow Q \vdash P \wedge Q \Rightarrow P \quad \frac{P \Rightarrow Q \vdash P \Rightarrow P \wedge Q}{P \Rightarrow Q \vdash P \wedge Q \Rightarrow P} \text{iffI}}{P \Rightarrow Q \vdash P \wedge Q \Leftrightarrow P} \\
 \frac{P \Rightarrow Q \vdash P \wedge Q \Rightarrow P \quad \frac{P \Rightarrow Q \vdash P \wedge Q \Rightarrow P}{P \Rightarrow Q \vdash P \wedge Q \Rightarrow P} \text{implI}}{P \Rightarrow Q \vdash P \wedge Q \Rightarrow P} \\
 \frac{P \Rightarrow Q \vdash P \wedge Q \Rightarrow P \quad \frac{P \Rightarrow Q \vdash P \wedge Q \Rightarrow P}{P \Rightarrow Q \vdash P \wedge Q \Rightarrow P} \text{rotate}}{P \Rightarrow Q \vdash P \wedge Q \Rightarrow P} \\
 \frac{P \Rightarrow Q \vdash P \wedge Q \Rightarrow P \quad \frac{P \Rightarrow Q \vdash P \wedge Q \Rightarrow P}{P \Rightarrow Q \vdash P \wedge Q \Rightarrow P} \text{impE}}{P \Rightarrow Q \vdash P \wedge Q \Rightarrow P} \\
 \frac{P \Rightarrow Q \vdash P \wedge Q \Rightarrow P \quad \frac{P \Rightarrow Q \vdash P \wedge Q \Rightarrow P}{P \Rightarrow Q \vdash P \wedge Q \Rightarrow P} \text{conjI}}{P \Rightarrow Q \vdash P \wedge Q \Rightarrow P} \\
 \frac{P \Rightarrow Q \vdash P \wedge Q \Rightarrow P \quad \frac{P \Rightarrow Q \vdash P \wedge Q \Rightarrow P}{P \Rightarrow Q \vdash P \wedge Q \Rightarrow P} \text{asm}}{P \Rightarrow Q \vdash P \wedge Q \Rightarrow P}
 \end{array}$$

Example: Rule of Subsumption

$$\begin{array}{c}
 \frac{P \Rightarrow Q \vdash P \wedge Q \Rightarrow P}{P \Rightarrow Q \vdash P \wedge Q \Leftrightarrow P} \\
 \frac{P \Rightarrow Q, P \vdash P \wedge Q \Rightarrow P}{P \Rightarrow Q \vdash P \Rightarrow P \wedge Q} \text{ iffI} \\
 \frac{P, P \Rightarrow Q \vdash P \wedge Q}{P \Rightarrow Q, P \vdash P \wedge Q} \text{ rotate} \\
 \frac{P \Rightarrow Q, P \vdash P \wedge Q}{P \Rightarrow Q, P \vdash P \wedge Q} \text{ rotate} \\
 \frac{P \Rightarrow Q, P \vdash P \wedge Q}{Q, P \vdash P \wedge Q} \text{ impE} \\
 \frac{P \vdash P}{Q, P \vdash P \wedge Q} \text{ conjI} \\
 \frac{P, Q \vdash P}{Q, P \vdash P} \text{ rotate} \\
 \frac{}{P, Q \vdash P} \text{ asm} \\
 \frac{Q, P \vdash P}{Q, P \vdash Q} \text{ conjI} \\
 \frac{}{Q, P \vdash Q} \text{ asm}
 \end{array}$$

Example: Rule of Subsumption

$$\begin{array}{c}
 \frac{P \Rightarrow Q \vdash P \wedge Q \Rightarrow P}{P \Rightarrow Q \vdash P \wedge Q \Leftrightarrow P} \\
 \frac{P \Rightarrow Q \vdash P \wedge Q \Rightarrow P}{P \Rightarrow Q \vdash P \Rightarrow P \wedge Q} \text{ iffI} \\
 \frac{P \Rightarrow Q \vdash P \Rightarrow P \wedge Q}{P \Rightarrow Q \vdash P \wedge Q} \text{ impI} \\
 \frac{P, P \Rightarrow Q \vdash P \wedge Q}{P \Rightarrow Q, P \vdash P \wedge Q} \text{ rotate} \\
 \frac{P \Rightarrow Q, P \vdash P \wedge Q}{Q, P \vdash P \wedge Q} \text{ impE} \\
 \frac{P \vdash P \quad Q, P \vdash P \wedge Q}{Q, P \vdash P} \text{ conjI} \\
 \frac{P \vdash P \quad Q, P \vdash P}{Q, P \vdash P} \text{ rotate} \\
 \frac{Q, P \vdash P}{P, Q \vdash P} \text{ asm} \\
 \frac{P, Q \vdash P}{P, Q \vdash P} \text{ asm}
 \end{array}$$

Natural Deduction Rules

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Example: Rule of Subsumption

$$\begin{array}{c}
 \frac{}{P \vdash P} \text{asm} \qquad \frac{}{P, Q \vdash P} \text{asm} \quad \frac{}{Q, P \vdash P} \text{rotate} \quad \frac{}{Q, P \vdash Q} \text{asm} \\
 \frac{}{P \vdash P} \text{asm} \quad \frac{}{Q, P \vdash P \wedge Q} \text{conjI} \\
 \frac{}{P \Rightarrow Q, P \vdash P \wedge Q} \text{impE} \\
 \frac{}{P, P \Rightarrow Q \vdash P \wedge Q} \text{rotate} \\
 \frac{}{P \Rightarrow Q \vdash P \Rightarrow P \wedge Q} \text{impI} \\
 \frac{}{P \Rightarrow Q \vdash P \wedge Q \Leftrightarrow P} \text{iffI}
 \end{array}$$

Example: Rule of Subsumption

$$\begin{array}{c}
 \frac{}{P \vdash P} \text{asm} \qquad \frac{}{P, Q \vdash P} \text{asm} \quad \frac{}{Q, P \vdash P} \text{rotate} \quad \frac{}{Q, P \vdash Q} \text{asm} \\
 \frac{}{P \vdash P} \text{asm} \quad \frac{}{Q, P \vdash P \wedge Q} \text{conjI} \\
 \frac{}{P \Rightarrow Q, P \vdash P \wedge Q} \text{impE} \\
 \frac{}{P \Rightarrow Q, P \vdash P \wedge Q} \text{rotate} \\
 \frac{}{P, P \Rightarrow Q \vdash P \wedge Q} \text{impI} \\
 \frac{}{P \Rightarrow Q \vdash P \Rightarrow P \wedge Q} \text{iffI} \\
 \frac{}{P \Rightarrow Q \vdash P \wedge Q \Leftrightarrow P} \text{impI}
 \end{array}$$

Example: Rule of Subsumption

$$\begin{array}{c}
 \frac{P \wedge Q, P \Rightarrow Q \vdash P}{P \Rightarrow Q \vdash P \wedge Q \Rightarrow P} \text{impI} \\
 \hline
 P \Rightarrow Q \vdash P \wedge Q \Leftrightarrow P
 \end{array}
 \qquad
 \begin{array}{c}
 \frac{-}{P \vdash P} \text{asm} \qquad
 \frac{\frac{-}{P, Q \vdash P} \text{asm}}{Q, P \vdash P} \text{rotate} \qquad
 \frac{-}{Q, P \vdash Q} \text{asm} \\
 \hline
 \frac{Q, P \vdash P \wedge Q}{Q, P \vdash P \wedge Q} \text{conjI} \\
 \hline
 \frac{P \Rightarrow Q, P \vdash P \wedge Q}{P, P \Rightarrow Q \vdash P \wedge Q} \text{impE} \\
 \hline
 \frac{P, P \Rightarrow Q \vdash P \wedge Q}{P \Rightarrow Q \vdash P \Rightarrow P \wedge Q} \text{rotate} \\
 \hline
 \frac{P \Rightarrow Q \vdash P \Rightarrow P \wedge Q}{P \Rightarrow Q \vdash P \wedge Q \Leftrightarrow P} \text{iffI}
 \end{array}$$

Example: Rule of Subsumption

$$\begin{array}{c}
 \frac{\frac{P \wedge Q, P \Rightarrow Q \vdash P}{P \Rightarrow Q \vdash P \wedge Q \Rightarrow P} \text{impI}}{P \Rightarrow Q \vdash P \wedge Q \Leftrightarrow P} \\
 \frac{\frac{\frac{\frac{-}{P \vdash P} \text{asm}}{P \Rightarrow Q, P \vdash P \wedge Q} \text{impE} \quad \frac{\frac{\frac{-}{P, Q \vdash P} \text{asm} \quad \frac{-}{Q, P \vdash P} \text{rotate}}{Q, P \vdash P \wedge Q} \text{conjI}}{P, P \Rightarrow Q \vdash P \wedge Q} \text{rotate}}{P \Rightarrow Q \vdash P \Rightarrow P \wedge Q} \text{impI}}{P \Rightarrow Q \vdash P \wedge Q \Leftrightarrow P} \text{iffI}
 \end{array}$$

Example: Rule of Subsumption

$$\begin{array}{c}
 \frac{\frac{P \wedge Q, P \Rightarrow Q \vdash P}{P \Rightarrow Q \vdash P \wedge Q \Rightarrow P} \text{conjE}}{P \Rightarrow Q \vdash P \wedge Q \Rightarrow P} \text{impI} \\
 \hline
 P \Rightarrow Q \vdash P \wedge Q \Leftrightarrow P
 \end{array}
 \qquad
 \begin{array}{c}
 \frac{\frac{\frac{-}{P \vdash P} \text{asm}}{Q, P \vdash P} \text{rotate} \quad \frac{\frac{-}{Q, P \vdash Q} \text{asm}}{Q, P \vdash P \wedge Q} \text{conjI}}{P \Rightarrow Q, P \vdash P \wedge Q} \text{impE} \\
 \frac{\frac{P \Rightarrow Q, P \vdash P \wedge Q}{P, P \Rightarrow Q \vdash P \wedge Q} \text{rotate}}{P \Rightarrow Q \vdash P \Rightarrow P \wedge Q} \text{impI} \\
 \hline
 P \Rightarrow Q \vdash P \Rightarrow P \wedge Q \text{ iffI}
 \end{array}$$

Example: Rule of Subsumption

$$\begin{array}{c}
 \frac{P, Q, P \Rightarrow Q \vdash P}{P \wedge Q, P \Rightarrow Q \vdash P} \text{conjE} \\
 \frac{P \wedge Q, P \Rightarrow Q \vdash P}{P \Rightarrow Q \vdash P \wedge Q \Rightarrow P} \text{impI} \\
 \hline
 P \Rightarrow Q \vdash P \wedge Q \Leftrightarrow P
 \end{array}
 \qquad
 \begin{array}{c}
 \frac{-}{P \vdash P} \text{asm} \\
 \frac{P, Q \vdash P}{Q, P \vdash P} \text{rotate} \\
 \frac{Q, P \vdash P \wedge Q}{P \Rightarrow Q, P \vdash P \wedge Q} \text{impE} \\
 \frac{P \Rightarrow Q, P \vdash P \wedge Q}{P, P \Rightarrow Q \vdash P \wedge Q} \text{rotate} \\
 \frac{P, P \Rightarrow Q \vdash P \wedge Q}{P \Rightarrow Q \vdash P \Rightarrow P \wedge Q} \text{impI} \\
 \hline
 P \Rightarrow Q \vdash P \Rightarrow P \wedge Q
 \end{array}
 \qquad
 \begin{array}{c}
 \frac{-}{Q, P \vdash Q} \text{asm} \\
 \frac{Q, P \vdash Q}{Q, P \vdash P} \text{conjI}
 \end{array}$$

Example: Rule of Subsumption

$$\begin{array}{c}
 \frac{\frac{\frac{P, Q, P \Rightarrow Q \vdash P}{P \wedge Q, P \Rightarrow Q \vdash P} \text{conjE}}{P \Rightarrow Q \vdash P \wedge Q \Rightarrow P} \text{impI}}{P \Rightarrow Q \vdash P \wedge Q \Leftrightarrow P} \\
 \\
 \frac{\frac{\frac{-}{P \vdash P} \text{asm}}{P \Rightarrow Q, P \vdash P \wedge Q} \text{impE}}{\frac{\frac{\frac{-}{P, Q \vdash P} \text{asm}}{Q, P \vdash P} \text{rotate}}{Q, P \vdash Q} \text{conjI}}{\frac{\frac{\frac{-}{P, P \Rightarrow Q \vdash P \wedge Q} \text{rotate}}{P, P \Rightarrow Q \vdash P \Rightarrow P \wedge Q} \text{impI}}{P \Rightarrow Q \vdash P \Rightarrow P \wedge Q} \text{iffI}}
 \end{array}$$

Example: Rule of Subsumption

$$\begin{array}{c}
 \frac{\frac{\frac{P, Q, P \Rightarrow Q \vdash P}{P \wedge Q, P \Rightarrow Q \vdash P} \text{asm}}{P \Rightarrow Q \vdash P \wedge Q \Rightarrow P} \text{conjE}}{P \Rightarrow Q \vdash P \wedge Q \Rightarrow P} \text{impI} \\
 \hline
 P \Rightarrow Q \vdash P \wedge Q \Leftrightarrow P
 \end{array}
 \qquad
 \begin{array}{c}
 \frac{-}{P \vdash P} \text{asm} \qquad
 \frac{\frac{-}{P, Q \vdash P} \text{asm}}{Q, P \vdash P} \text{rotate} \qquad
 \frac{-}{Q, P \vdash Q} \text{asm} \\
 \hline
 \frac{Q, P \vdash P \wedge Q}{P \Rightarrow Q, P \vdash P \wedge Q} \text{conjI} \\
 \hline
 \frac{P \Rightarrow Q, P \vdash P \wedge Q}{P, P \Rightarrow Q \vdash P \wedge Q} \text{impE} \\
 \hline
 \frac{P, P \Rightarrow Q \vdash P \wedge Q}{P \Rightarrow Q \vdash P \Rightarrow P \wedge Q} \text{rotate} \\
 \hline
 \frac{P \Rightarrow Q \vdash P \Rightarrow P \wedge Q}{P \Rightarrow Q \vdash P \wedge Q \Leftrightarrow P} \text{iffI}
 \end{array}$$

Example: Rule of Subsumption

$$\begin{array}{c}
 \frac{}{P, Q, P \Rightarrow Q \vdash P} \text{asm} \\
 \frac{}{P \wedge Q, P \Rightarrow Q \vdash P} \text{conjE} \\
 \frac{}{P \Rightarrow Q \vdash P \wedge Q \Rightarrow P} \text{impI} \\
 \hline
 P \Rightarrow Q \vdash P \wedge Q \Leftrightarrow P
 \end{array}
 \qquad
 \begin{array}{c}
 \frac{}{P \vdash P} \text{asm} \\
 \frac{}{P \Rightarrow Q, P \vdash P \wedge Q} \text{rotate} \\
 \frac{}{P, P \Rightarrow Q \vdash P \wedge Q} \text{rotate} \\
 \frac{}{P \Rightarrow Q \vdash P \Rightarrow P \wedge Q} \text{impI} \\
 \hline
 P \Rightarrow Q \vdash P \Rightarrow P \wedge Q
 \end{array}
 \qquad
 \begin{array}{c}
 \frac{}{P, Q \vdash P} \text{asm} \\
 \frac{}{Q, P \vdash P} \text{rotate} \\
 \frac{}{Q, P \vdash Q} \text{asm} \\
 \frac{}{Q, P \vdash P \wedge Q} \text{conjI} \\
 \hline
 Q, P \vdash P \wedge Q
 \end{array}
 \qquad
 \begin{array}{c}
 \frac{}{P \Rightarrow Q, P \vdash P \wedge Q} \text{impE} \\
 \frac{}{P, P \Rightarrow Q \vdash P \wedge Q} \text{rotate} \\
 \frac{}{P \Rightarrow Q \vdash P \Rightarrow P \wedge Q} \text{impI} \\
 \hline
 P \Rightarrow Q \vdash P \Rightarrow P \wedge Q
 \end{array}$$

Outline

Natural Deduction Rules

Introduction

Basic Rules

Conjunction

Disjunction

Implication

Equivalence

Negation

Summary

Negation

- ▶ The rule for negation introduction is used to prove $\Gamma \vdash \neg P$.
- ▶ The proof proceeds by assuming P and then showing a contradiction:

$$\frac{P, \Gamma \vdash \text{false}}{\Gamma \vdash \neg P} \text{notI}$$

- ▶ The rule for classical contradiction mirrors this and is used to prove $\Gamma \vdash P$.
- ▶ The proof proceeds by assuming $\neg P$ and then showing a contradiction:

$$\frac{\neg P, \Gamma \vdash \text{false}}{\Gamma \vdash P} \text{notE}$$

- ▶ Negation elimination is used with a hypothesis $\neg P$.
- ▶ The proof proceeds by proving the contradiction P :

$$\frac{\Gamma \vdash P \quad \neg P \in \Gamma}{\text{false}} \text{notE}$$

Negation

- ▶ The rule for negation introduction is used to prove $\Gamma \vdash \neg P$.
- ▶ The proof proceeds by assuming P and then showing a contradiction:

$$\frac{P, \Gamma \vdash \text{false}}{\Gamma \vdash \neg P} \text{notI}$$

- ▶ The rule for classical contradiction mirrors this and is used to prove $\Gamma \vdash P$.
- ▶ The proof proceeds by assuming $\neg P$ and then showing a contradiction:

$$\frac{\neg P, \Gamma \vdash \text{false}}{\Gamma \vdash P} \text{notE}$$

- ▶ Negation elimination is used with a hypothesis $\neg P$.
- ▶ The proof proceeds by proving the contradiction P :

$$\frac{\Gamma \vdash P}{\neg P, \Gamma \vdash \text{false}} \text{notE}$$

Negation

- ▶ The rule for negation introduction is used to prove $\Gamma \vdash \neg P$.
- ▶ The proof proceeds by assuming P and then showing a contradiction:

$$\frac{P, \Gamma \vdash \text{false}}{\Gamma \vdash \neg P} \text{notI}$$

- ▶ The rule for classical contradiction mirrors this and is used to prove $\Gamma \vdash P$.
- ▶ The proof proceeds by assuming $\neg P$ and then showing a contradiction:

$$\frac{\neg P, \Gamma \vdash \text{false}}{\Gamma \vdash P} \text{notE}$$

- ▶ Negation elimination is used with a hypothesis $\neg P$.
- ▶ The proof proceeds by proving the contradiction P :

$$\frac{\Gamma \vdash P \quad \Gamma \vdash \neg P}{\Gamma \vdash \text{false}} \text{notE}$$

Negation

- ▶ The rule for negation introduction is used to prove $\Gamma \vdash \neg P$.
- ▶ The proof proceeds by assuming P and then showing a contradiction:

$$\frac{P, \Gamma \vdash \text{false}}{\Gamma \vdash \neg P} \text{notI}$$

- ▶ The rule for classical contradiction mirrors this and is used to prove $\Gamma \vdash P$.
- ▶ The proof proceeds by assuming $\neg P$ and then showing a contradiction:

$$\frac{\neg P, \Gamma \vdash \text{false}}{\Gamma \vdash P} \text{contr}$$

- ▶ Negation elimination is used with a hypothesis $\neg P$.
- ▶ The proof proceeds by proving the contradiction P :

$$\frac{\Gamma \vdash P \quad \Gamma \vdash \neg P}{\Gamma \vdash \text{false}} \text{notE}$$

Negation

- ▶ The rule for negation introduction is used to prove $\Gamma \vdash \neg P$.
- ▶ The proof proceeds by assuming P and then showing a contradiction:

$$\frac{P, \Gamma \vdash \text{false}}{\Gamma \vdash \neg P} \text{notI}$$

- ▶ The rule for classical contradiction mirrors this and is used to prove $\Gamma \vdash P$.
- ▶ The proof proceeds by assuming $\neg P$ and then showing a contradiction:

$$\frac{\neg P, \Gamma \vdash \text{false}}{\Gamma \vdash P} \text{ccontr}$$

- ▶ Negation elimination is used with a hypothesis $\neg P$.
- ▶ The proof proceeds by proving the contradiction P :

$$\frac{\Gamma \vdash P \quad \Gamma \vdash \neg P}{\Gamma \vdash \text{false}} \text{notE}$$

Negation

- ▶ The rule for negation introduction is used to prove $\Gamma \vdash \neg P$.
- ▶ The proof proceeds by assuming P and then showing a contradiction:

$$\frac{P, \Gamma \vdash \text{false}}{\Gamma \vdash \neg P} \text{notI}$$

- ▶ The rule for classical contradiction mirrors this and is used to prove $\Gamma \vdash P$.
- ▶ The proof proceeds by assuming $\neg P$ and then showing a contradiction:

$$\frac{\neg P, \Gamma \vdash \text{false}}{\Gamma \vdash P} \text{ccontr}$$

- ▶ Negation elimination is used with a hypothesis $\neg P$.
- ▶ The proof proceeds by proving the contradiction P :

Negation

- ▶ The rule for negation introduction is used to prove $\Gamma \vdash \neg P$.
- ▶ The proof proceeds by assuming P and then showing a contradiction:

$$\frac{P, \Gamma \vdash \text{false}}{\Gamma \vdash \neg P} \text{notI}$$

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Example: Proof with Negation

Natural Deduction Rules

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- ▶ Consider the sequent $\neg (P \vee Q) \vdash \neg Q$.
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Natural Deduction Rules

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- ▶ The following disjunction rule is *safe*:

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- ▶ The Law of Excluded Middle states that for every P , either P or $\neg P$ is true.
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- ▶ This follows directly from the classical disjunction introduction rule.
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Soundness and Completeness

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An axiom is an inference rule with empty premises.

- ▶ Definition

A derivation is tree of propositions labelled by rule names.

- ▶ Definition

A derivation is a proof if all its leaves are axioms.

- ▶ Theorem (Soundness)

The root sequent of a proof is valid.

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If a sequent is valid, then there is a proof with that sequent as root.

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Outline

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Basic Rules

Conjunction

Disjunction

Implication

Equivalence

Negation

Summary

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- ▶ We presented the Natural Deduction Laws for propositions.
- ▶ Sound and complete system for reasoning about propositional calculus.
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