Natural Deduction
Predicate Calculus

Simon Foster & Jim Woodcock

University of York

PROF

27th November 2021

Overview

Natural Deduction Rules

Universal Quantification

Existential Quantification

Outline

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Universal Quantification

Existential Quantification

Natural Deduction Rules: Predicate Calculus

$$x \notin fv(\Gamma) \; \frac{\Gamma \vdash P(x)}{\Gamma \vdash \forall\, a \bullet P(a)} \; \texttt{allI} \qquad \frac{P(t), \Gamma \vdash R}{\forall\, a \bullet P(a), \Gamma \vdash R} \; \texttt{allE}$$

$$\frac{\Gamma \vdash P(t)}{\Gamma \vdash \exists\, a \bullet P(a)} \; \texttt{exI} \qquad x \notin fv(\Gamma, Q) \; \frac{P(x), \Gamma \vdash Q}{\exists\, a \bullet P(a), \Gamma \vdash Q} \; \texttt{exE}$$

Reasoning with Universal Quantification

- How can we reason with the universal quantifier?
- Take inspiration from the universal quantifier as generalised conjunction.
- Use and-introduction: $\dfrac{\Gamma \vdash P \qquad \Gamma \vdash Q}{\Gamma \vdash P \wedge Q}$ `conjI`.
- But this needs an infinite number of proofs to complete the proof tree!
- Insight: just prove a typical instance $P(i)$ then generalise to all instances.
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Reasoning with Universal Quantification

- We formalise the notion of an arbitrary choice.
- We forbid the bound variable being mentioned in any hypothesis: $x \notin fv(\Gamma)$.
- So we are making no formal assumption about x.
- Universal introduction:

$$x \notin fv(\Gamma)\ \frac{\Gamma \vdash P(x)}{\Gamma \vdash \forall a \bullet P(a)}\ \texttt{allI}$$

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Universal Elimination

- How do we deal with universal quantification as a hypothesis?
- Again, we take inspiration from the rule for conjunction.
- Use and-elimination: $\dfrac{P, Q, \Gamma \vdash R}{P \land Q, \Gamma \vdash R}\,\texttt{conjE}$
- But this adds an infinite number of hypotheses!
- Insight: Instead of adding all instances of $P(i)$, add a specific one: $P(t)$.
- Specialisation (contrast with generalisation).

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$$\dfrac{\dfrac{y \notin fv(P(x)) \quad \dfrac{P(x) \vdash P(y) \vee Q(y)}{P(x) \vdash \forall a \bullet P(a) \vee Q(a)}\ \texttt{allI}}{\forall a \bullet P(a) \vdash \forall a \bullet P(a) \vee Q(a)}\ \texttt{allE}}{\vdash (\forall a \bullet P(a)) \Rightarrow (\forall a \bullet P(a) \vee Q(a))}\ \texttt{impI}$$

$$\dfrac{x \notin fv(\forall a \bullet P(a)) \quad \dfrac{\dfrac{\dfrac{\dfrac{}{P(x) \vdash P(x)}}{\forall a \bullet P(a) \vdash P(x)}\ \texttt{allE}}{\forall a \bullet P(a) \vdash P(x) \vee Q(x)}\ \texttt{disjI1}}{\forall a \bullet P(a) \vdash \forall a \bullet P(a) \vee Q(a)}\ \texttt{allI}}{\vdash (\forall a \bullet P(a)) \Rightarrow (\forall a \bullet P(a) \vee Q(a))}\ \texttt{impI}$$

Example: Reasoning with Universal Quantification

- Let's prove that $(\forall\, a \bullet P(a)) \Rightarrow (\forall\, a \bullet P(a) \vee Q(a))$.
- This proof uses both universal quantification rules.

$$
\cfrac{\cfrac{y \notin fv(P(x)) \quad \cfrac{P(x) \vdash P(y) \vee Q(y)}{P(x) \vdash \forall\, a \bullet P(a) \vee Q(a)}\ \mathtt{allI}}{\forall\, a \bullet P(a) \vdash \forall\, a \bullet P(a) \vee Q(a)}\ \mathtt{allE}}{\vdash (\forall\, a \bullet P(a)) \Rightarrow (\forall\, a \bullet P(a) \vee Q(a))}\ \mathtt{impI}
$$

$$
\cfrac{x \notin fv(\forall\, a \bullet P(a)) \quad \cfrac{\cfrac{\cfrac{\cfrac{}{P(x) \vdash P(x)}\ \mathtt{asm}}{\forall\, a \bullet P(a) \vdash P(x)}\ \mathtt{allE}}{\forall\, a \bullet P(a) \vdash P(x) \vee Q(x)}\ \mathtt{disjI1}}{\forall\, a \bullet P(a) \vdash \forall\, a \bullet P(a) \vee Q(a)}\ \mathtt{allI}}{\vdash (\forall\, a \bullet P(a)) \Rightarrow (\forall\, a \bullet P(a) \vee Q(a))}\ \mathtt{impI}
$$

Example: Reasoning with Universal Quantification

- Let's prove that $(\forall\, a \bullet P(a)) \Rightarrow (\forall\, a \bullet P(a) \vee Q(a))$.
- This proof uses both universal quantification rules.

$$
\dfrac{\dfrac{y \notin fv(P(x)) \quad \dfrac{P(x) \vdash P(y) \vee Q(y)}{P(x) \vdash \forall\, a \bullet P(a) \vee Q(a)}\ \texttt{allI}}{\forall\, a \bullet P(a) \vdash \forall\, a \bullet P(a) \vee Q(a)}\ \texttt{allE}}{\vdash (\forall\, a \bullet P(a)) \Rightarrow (\forall\, a \bullet P(a) \vee Q(a))}\ \texttt{impI}
$$

$$
\dfrac{x \notin fv(\forall\, a \bullet P(a)) \quad \dfrac{\dfrac{\dfrac{\dfrac{-}{P(x) \vdash P(x)}\ \texttt{asm}}{\forall\, a \bullet P(a) \vdash P(x)}\ \texttt{allE}}{\forall\, a \bullet P(a) \vdash P(x) \vee Q(x)}\ \texttt{disjI1}}{\forall\, a \bullet P(a) \vdash \forall\, a \bullet P(a) \vee Q(a)}\ \texttt{allI}}{\vdash (\forall\, a \bullet P(a)) \Rightarrow (\forall\, a \bullet P(a) \vee Q(a))}\ \texttt{impI}
$$

Example: Universal Quantification Distributes through Conjunction (L2R)

$$(\forall a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))$$

Proof

Example: Universal Quantification Distributes through Conjunction (L2R)

$$(\forall\, a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall\, a \bullet P(a)) \wedge (\forall\, a \bullet Q(a))$$

Proof

Example: Universal Quantification Distributes through Conjunction (L2R)

Natural Deduction Rules

$$(\forall a \bullet P(a) \land Q(a)) \Rightarrow (\forall a \bullet P(a)) \land (\forall a \bullet Q(a))$$

Proof

Example: Universal Quantification Distributes through Conjunction (L2R)

Natural Deduction Rules

$$(\forall\, a \bullet P(a) \land Q(a)) \Rightarrow (\forall\, a \bullet P(a)) \land (\forall\, a \bullet Q(a))$$

Proof

$$\vdash (\forall\, a \bullet P(a) \land Q(a)) \Rightarrow (\forall\, a \bullet P(a)) \land (\forall\, a \bullet Q(a))$$

Example: Universal Quantification Distributes through Conjunction (L2R)

Natural Deduction Rules

$$(\forall\, a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall\, a \bullet P(a)) \wedge (\forall\, a \bullet Q(a))$$

Proof

$$\frac{}{\vdash (\forall\, a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall\, a \bullet P(a)) \wedge (\forall\, a \bullet Q(a))}$$

Example: Universal Quantification Distributes through Conjunction (L2R)

Natural Deduction Rules

$$(\forall\, a \bullet P(a) \land Q(a)) \Rightarrow (\forall\, a \bullet P(a)) \land (\forall\, a \bullet Q(a))$$

Proof

$$\frac{}{\vdash (\forall\, a \bullet P(a) \land Q(a)) \Rightarrow (\forall\, a \bullet P(a)) \land (\forall\, a \bullet Q(a))}\ \texttt{implI}$$

Example: Universal Quantification Distributes through Conjunction (L2R)

Natural Deduction Rules

$$(\forall\, a \bullet P(a) \land Q(a)) \Rightarrow (\forall\, a \bullet P(a)) \land (\forall\, a \bullet Q(a))$$

Proof

$$\dfrac{\forall\, a \bullet P(a) \land Q(a) \vdash (\forall\, a \bullet P(a)) \land (\forall\, a \bullet Q(a))}{\vdash (\forall\, a \bullet P(a) \land Q(a)) \Rightarrow (\forall\, a \bullet P(a)) \land (\forall\, a \bullet Q(a))}\ \texttt{implI}$$

Example: Universal Quantification Distributes through Conjunction (L2R)

Natural Deduction Rules

$$(\forall a \bullet P(a) \land Q(a)) \Rightarrow (\forall a \bullet P(a)) \land (\forall a \bullet Q(a))$$

Proof

$$\dfrac{\dfrac{}{\forall a \bullet P(a) \land Q(a) \vdash (\forall a \bullet P(a)) \land (\forall a \bullet Q(a))}}{\vdash (\forall a \bullet P(a) \land Q(a)) \Rightarrow (\forall a \bullet P(a)) \land (\forall a \bullet Q(a))}\ \mathtt{implI}$$

Example: Universal Quantification Distributes through Conjunction (L2R)

Natural Deduction Rules

$$(\forall\, a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall\, a \bullet P(a)) \wedge (\forall\, a \bullet Q(a))$$

Proof

$$\dfrac{\dfrac{}{\forall\, a \bullet P(a) \wedge Q(a) \vdash (\forall\, a \bullet P(a)) \wedge (\forall\, a \bullet Q(a))}\;\texttt{conjI}}{\vdash (\forall\, a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall\, a \bullet P(a)) \wedge (\forall\, a \bullet Q(a))}\;\texttt{implI}$$

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Natural Deduction Rules

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Proof

$$\dfrac{\dfrac{\forall a \bullet P(a) \land Q(a) \vdash \forall a \bullet P(a) \qquad \forall a \bullet P(a) \land Q(a) \vdash \forall a \bullet Q(a)}{\forall a \bullet P(a) \land Q(a) \vdash (\forall a \bullet P(a)) \land (\forall a \bullet Q(a))}\ \texttt{conjI}}{\vdash (\forall a \bullet P(a) \land Q(a)) \Rightarrow (\forall a \bullet P(a)) \land (\forall a \bullet Q(a))}\ \texttt{implI}$$

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Proof

$$\dfrac{\dfrac{\forall\, a \bullet P(a) \land Q(a) \vdash \forall\, a \bullet P(a) \qquad \dfrac{}{\forall\, a \bullet P(a) \land Q(a) \vdash \forall\, a \bullet Q(a)}}{\forall\, a \bullet P(a) \land Q(a) \vdash (\forall\, a \bullet P(a)) \land (\forall\, a \bullet Q(a))}\ \texttt{conjI}}{\vdash (\forall\, a \bullet P(a) \land Q(a)) \Rightarrow (\forall\, a \bullet P(a)) \land (\forall\, a \bullet Q(a))}\ \texttt{implI}$$

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Proof

$$\dfrac{\dfrac{\forall\, a \bullet P(a) \land Q(a) \vdash \forall\, a \bullet P(a) \qquad \dfrac{}{\forall\, a \bullet P(a) \land Q(a) \vdash \forall\, a \bullet Q(a)}\,\texttt{allI}}{\forall\, a \bullet P(a) \land Q(a) \vdash (\forall\, a \bullet P(a)) \land (\forall\, a \bullet Q(a))}\,\texttt{conjI}}{\vdash (\forall\, a \bullet P(a) \land Q(a)) \Rightarrow (\forall\, a \bullet P(a)) \land (\forall\, a \bullet Q(a))}\,\texttt{implI}$$

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$$(\forall\, a \bullet P(a) \land Q(a)) \Rightarrow (\forall\, a \bullet P(a)) \land (\forall\, a \bullet Q(a))$$

Proof

$$\dfrac{\dfrac{\forall\, a \bullet P(a) \land Q(a) \vdash \forall\, a \bullet P(a) \qquad x \notin fv(\Gamma_1)\ \dfrac{}{\forall\, a \bullet P(a) \land Q(a) \vdash \forall\, a \bullet Q(a)}\ \texttt{allI}}{\forall\, a \bullet P(a) \land Q(a) \vdash (\forall\, a \bullet P(a)) \land (\forall\, a \bullet Q(a))}\ \texttt{conjI}}{\vdash (\forall\, a \bullet P(a) \land Q(a)) \Rightarrow (\forall\, a \bullet P(a)) \land (\forall\, a \bullet Q(a))}\ \texttt{implI}$$

Example: Universal Quantification Distributes through Conjunction (L2R)

Natural Deduction Rules

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Proof

$$\dfrac{\forall a \bullet P(a) \land Q(a) \vdash \forall a \bullet P(a) \qquad x \notin fv(\Gamma_1)\ \dfrac{\forall a \bullet P(a) \land Q(a) \vdash Q(x)}{\forall a \bullet P(a) \land Q(a) \vdash \forall a \bullet Q(a)}\ \texttt{allI}}{\dfrac{\forall a \bullet P(a) \land Q(a) \vdash (\forall a \bullet P(a)) \land (\forall a \bullet Q(a))}{\vdash (\forall a \bullet P(a) \land Q(a)) \Rightarrow (\forall a \bullet P(a)) \land (\forall a \bullet Q(a))}\ \texttt{implI}}\ \texttt{conjI}$$

Example: Universal Quantification Distributes through Conjunction (L2R)

Natural Deduction Rules

$$(\forall\, a \bullet P(a) \land Q(a)) \Rightarrow (\forall\, a \bullet P(a)) \land (\forall\, a \bullet Q(a))$$

Proof

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Example: Universal Quantification Distributes through Conjunction (L2R)

$$(\forall\, a \bullet P(a) \land Q(a)) \Rightarrow (\forall\, a \bullet P(a)) \land (\forall\, a \bullet Q(a))$$

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Example: Universal Quantification Distributes through Conjunction (L2R)

Natural Deduction Rules

$$(\forall\, a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall\, a \bullet P(a)) \wedge (\forall\, a \bullet Q(a))$$

Proof

$$\dfrac{\forall\, a \bullet P(a) \wedge Q(a) \vdash \forall\, a \bullet P(a) \qquad x \notin fv(\Gamma_1)\ \dfrac{\dfrac{P(x) \wedge Q(x) \vdash Q(x)}{\forall\, a \bullet P(a) \wedge Q(a) \vdash Q(x)}\ \texttt{allE}}{\forall\, a \bullet P(a) \wedge Q(a) \vdash \forall\, a \bullet Q(a)}\ \texttt{allI}}{\dfrac{\forall\, a \bullet P(a) \wedge Q(a) \vdash (\forall\, a \bullet P(a)) \wedge (\forall\, a \bullet Q(a))}{\vdash (\forall\, a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall\, a \bullet P(a)) \wedge (\forall\, a \bullet Q(a))}\ \texttt{implI}}\ \texttt{conjI}$$

Example: Universal Quantification Distributes through Conjunction (L2R)

$$(\forall\, a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall\, a \bullet P(a)) \wedge (\forall\, a \bullet Q(a))$$

Proof

$$\dfrac{\dfrac{\forall\, a \bullet P(a) \wedge Q(a) \vdash \forall\, a \bullet P(a) \qquad x \notin fv(\Gamma_1)\ \dfrac{\dfrac{\dfrac{}{P(x) \wedge Q(x) \vdash Q(x)}}{\forall\, a \bullet P(a) \wedge Q(a) \vdash Q(x)}\ \texttt{allE}}{\forall\, a \bullet P(a) \wedge Q(a) \vdash \forall\, a \bullet Q(a)}\ \texttt{allI}}{\forall\, a \bullet P(a) \wedge Q(a) \vdash (\forall\, a \bullet P(a)) \wedge (\forall\, a \bullet Q(a))}\ \texttt{conjI}}{\vdash (\forall\, a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall\, a \bullet P(a)) \wedge (\forall\, a \bullet Q(a))}\ \texttt{implI}$$

Example: Universal Quantification Distributes through Conjunction (L2R)

Natural Deduction Rules

$$(\forall\, a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall\, a \bullet P(a)) \wedge (\forall\, a \bullet Q(a))$$

Proof

$$\dfrac{\forall\, a \bullet P(a) \wedge Q(a) \vdash \forall\, a \bullet P(a) \qquad x \notin fv(\Gamma_1)\ \dfrac{\dfrac{\dfrac{}{P(x) \wedge Q(x) \vdash Q(x)}\ \texttt{conjE}}{\forall\, a \bullet P(a) \wedge Q(a) \vdash Q(x)}\ \texttt{allE}}{\forall\, a \bullet P(a) \wedge Q(a) \vdash \forall\, a \bullet Q(a)}\ \texttt{allI}}{\dfrac{\forall\, a \bullet P(a) \wedge Q(a) \vdash (\forall\, a \bullet P(a)) \wedge (\forall\, a \bullet Q(a))}{\vdash (\forall\, a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall\, a \bullet P(a)) \wedge (\forall\, a \bullet Q(a))}\ \texttt{implI}}\ \texttt{conjI}$$

Example: Universal Quantification Distributes through Conjunction (L2R)

Natural Deduction Rules

$$(\forall a \bullet P(a) \land Q(a)) \Rightarrow (\forall a \bullet P(a)) \land (\forall a \bullet Q(a))$$

Proof

$$
\begin{prooftree}
\AxiomC{$\forall a \bullet P(a) \land Q(a) \vdash \forall a \bullet P(a)$}
\AxiomC{$P(x), Q(x) \vdash Q(x)$}
\RightLabel{$\texttt{conjE}$}
\UnaryInfC{$P(x) \land Q(x) \vdash Q(x)$}
\RightLabel{$\texttt{allE}$}
\UnaryInfC{$\forall a \bullet P(a) \land Q(a) \vdash Q(x)$}
\LeftLabel{$x \notin fv(\Gamma_1)$}
\RightLabel{$\texttt{allI}$}
\UnaryInfC{$\forall a \bullet P(a) \land Q(a) \vdash \forall a \bullet Q(a)$}
\RightLabel{$\texttt{conjI}$}
\BinaryInfC{$\forall a \bullet P(a) \land Q(a) \vdash (\forall a \bullet P(a)) \land (\forall a \bullet Q(a))$}
\RightLabel{$\texttt{implI}$}
\UnaryInfC{$\vdash (\forall a \bullet P(a) \land Q(a)) \Rightarrow (\forall a \bullet P(a)) \land (\forall a \bullet Q(a))$}
\end{prooftree}
$$

Example: Universal Quantification Distributes through Conjunction (L2R)

Natural Deduction Rules

$$(\forall a \bullet P(a) \land Q(a)) \Rightarrow (\forall a \bullet P(a)) \land (\forall a \bullet Q(a))$$

Proof

$$
\dfrac{
  \forall a \bullet P(a) \land Q(a) \vdash \forall a \bullet P(a)
  \qquad
  {\scriptstyle x \notin fv(\Gamma_1)}\
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{}{P(x), Q(x) \vdash Q(x)}
      }{P(x) \land Q(x) \vdash Q(x)}\ \mathtt{conjE}
    }{\forall a \bullet P(a) \land Q(a) \vdash Q(x)}\ \mathtt{allE}
  }{\forall a \bullet P(a) \land Q(a) \vdash \forall a \bullet Q(a)}\ \mathtt{allI}
}{
  \dfrac{\forall a \bullet P(a) \land Q(a) \vdash (\forall a \bullet P(a)) \land (\forall a \bullet Q(a))}{\vdash (\forall a \bullet P(a) \land Q(a)) \Rightarrow (\forall a \bullet P(a)) \land (\forall a \bullet Q(a))}\ \mathtt{implI}
}\ \mathtt{conjI}
$$

Example: Universal Quantification Distributes through Conjunction (L2R)

Natural Deduction Rules

$$(\forall\, a \bullet P(a) \land Q(a)) \Rightarrow (\forall\, a \bullet P(a)) \land (\forall\, a \bullet Q(a))$$

Proof

$$\dfrac{\forall\, a \bullet P(a) \land Q(a) \vdash \forall\, a \bullet P(a) \qquad x \notin fv(\Gamma_1)\ \dfrac{\dfrac{\dfrac{\dfrac{}{P(x), Q(x) \vdash Q(x)}\ \mathtt{thin}}{P(x) \land Q(x) \vdash Q(x)}\ \mathtt{conjE}}{\forall\, a \bullet P(a) \land Q(a) \vdash Q(x)}\ \mathtt{allE}}{\forall\, a \bullet P(a) \land Q(a) \vdash \forall\, a \bullet Q(a)}\ \mathtt{allI}}{\dfrac{\forall\, a \bullet P(a) \land Q(a) \vdash (\forall\, a \bullet P(a)) \land (\forall\, a \bullet Q(a))}{\vdash (\forall\, a \bullet P(a) \land Q(a)) \Rightarrow (\forall\, a \bullet P(a)) \land (\forall\, a \bullet Q(a))}\ \mathtt{implI}}\ \mathtt{conjI}$$

Example: Universal Quantification Distributes through Conjunction (L2R)

Natural Deduction Rules

$$(\forall a \bullet P(a) \land Q(a)) \Rightarrow (\forall a \bullet P(a)) \land (\forall a \bullet Q(a))$$

Proof

$$\dfrac{\dfrac{\forall a \bullet P(a) \land Q(a) \vdash \forall a \bullet P(a) \qquad {\scriptstyle x \notin fv(\Gamma_1)}\ \dfrac{\dfrac{\dfrac{\dfrac{Q(x) \vdash Q(x)}{P(x), Q(x) \vdash Q(x)}\ \texttt{thin}}{P(x) \land Q(x) \vdash Q(x)}\ \texttt{conjE}}{\forall a \bullet P(a) \land Q(a) \vdash Q(x)}\ \texttt{allE}}{\forall a \bullet P(a) \land Q(a) \vdash \forall a \bullet Q(a)}\ \texttt{allI}}{\forall a \bullet P(a) \land Q(a) \vdash (\forall a \bullet P(a)) \land (\forall a \bullet Q(a))}\ \texttt{conjI}}{\vdash (\forall a \bullet P(a) \land Q(a)) \Rightarrow (\forall a \bullet P(a)) \land (\forall a \bullet Q(a))}\ \texttt{implI}$$

Example: Universal Quantification Distributes through Conjunction (L2R)

Natural Deduction Rules

$$(\forall\, a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall\, a \bullet P(a)) \wedge (\forall\, a \bullet Q(a))$$

Proof

$$
\begin{prooftree}
\AxiomC{$\forall\, a \bullet P(a) \wedge Q(a) \vdash \forall\, a \bullet P(a)$}
\AxiomC{}
\UnaryInfC{$Q(x) \vdash Q(x)$}
\RightLabel{thin}
\UnaryInfC{$P(x), Q(x) \vdash Q(x)$}
\RightLabel{conjE}
\UnaryInfC{$P(x) \wedge Q(x) \vdash Q(x)$}
\RightLabel{allE}
\UnaryInfC{$\forall\, a \bullet P(a) \wedge Q(a) \vdash Q(x)$}
\LeftLabel{$x \notin fv(\Gamma_1)$}
\RightLabel{allI}
\UnaryInfC{$\forall\, a \bullet P(a) \wedge Q(a) \vdash \forall\, a \bullet Q(a)$}
\RightLabel{conjI}
\BinaryInfC{$\forall\, a \bullet P(a) \wedge Q(a) \vdash (\forall\, a \bullet P(a)) \wedge (\forall\, a \bullet Q(a))$}
\RightLabel{implI}
\UnaryInfC{$\vdash (\forall\, a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall\, a \bullet P(a)) \wedge (\forall\, a \bullet Q(a))$}
\end{prooftree}
$$

Example: Universal Quantification Distributes through Conjunction (L2R)

$$(\forall a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))$$

Proof

$$\dfrac{\forall a \bullet P(a) \wedge Q(a) \vdash \forall a \bullet P(a) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{Q(x) \vdash Q(x)}\ \texttt{asm}}{P(x), Q(x) \vdash Q(x)}\ \texttt{thin}}{P(x) \wedge Q(x) \vdash Q(x)}\ \texttt{conjE}}{\forall a \bullet P(a) \wedge Q(a) \vdash Q(x)}\ \texttt{allE}}{\forall a \bullet P(a) \wedge Q(a) \vdash \forall a \bullet Q(a)}\ \texttt{allI}\ \ x \notin fv(\Gamma_1)}{\dfrac{\forall a \bullet P(a) \wedge Q(a) \vdash (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))}{\vdash (\forall a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))}\ \texttt{implI}}\ \texttt{conjI}$$

Example: Universal Quantification Distributes through Conjunction (L2R)

Natural Deduction Rules

$$(\forall a \bullet P(a) \land Q(a)) \Rightarrow (\forall a \bullet P(a)) \land (\forall a \bullet Q(a))$$

Proof

$$\dfrac{\forall a \bullet P(a) \land Q(a) \vdash \forall a \bullet P(a) \qquad x \notin fv(\Gamma_1)\;\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{-}{Q(x) \vdash Q(x)}\,\texttt{asm}}{P(x), Q(x) \vdash Q(x)}\,\texttt{thin}}{P(x) \land Q(x) \vdash Q(x)}\,\texttt{conjE}}{\forall a \bullet P(a) \land Q(a) \vdash Q(x)}\,\texttt{allE}}{\forall a \bullet P(a) \land Q(a) \vdash \forall a \bullet Q(a)}\,\texttt{allI}}{\dfrac{\forall a \bullet P(a) \land Q(a) \vdash (\forall a \bullet P(a)) \land (\forall a \bullet Q(a))}{\vdash (\forall a \bullet P(a) \land Q(a)) \Rightarrow (\forall a \bullet P(a)) \land (\forall a \bullet Q(a))}\,\texttt{implI}}\,\texttt{conjI}$$

Example: Universal Quantification Distributes through Conjunction (L2R)

Natural Deduction Rules

$$(\forall\, a \bullet P(a) \land Q(a)) \Rightarrow (\forall\, a \bullet P(a)) \land (\forall\, a \bullet Q(a))$$

Proof

$$\dfrac{\dfrac{\overline{\forall\, a \bullet P(a) \land Q(a) \vdash \forall\, a \bullet P(a)} \qquad x \notin fv(\Gamma_1)\ \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{-}{Q(x) \vdash Q(x)}\ \texttt{asm}}{P(x), Q(x) \vdash Q(x)}\ \texttt{thin}}{P(x) \land Q(x) \vdash Q(x)}\ \texttt{conjE}}{\forall\, a \bullet P(a) \land Q(a) \vdash Q(x)}\ \texttt{allE}}{\forall\, a \bullet P(a) \land Q(a) \vdash \forall\, a \bullet Q(a)}\ \texttt{allI}}{\forall\, a \bullet P(a) \land Q(a) \vdash (\forall\, a \bullet P(a)) \land (\forall\, a \bullet Q(a))}\ \texttt{conjI}}{\vdash (\forall\, a \bullet P(a) \land Q(a)) \Rightarrow (\forall\, a \bullet P(a)) \land (\forall\, a \bullet Q(a))}\ \texttt{implI}$$

Example: Universal Quantification Distributes through Conjunction (L2R)

Natural Deduction Rules

$$(\forall\, a \bullet P(a) \land Q(a)) \Rightarrow (\forall\, a \bullet P(a)) \land (\forall\, a \bullet Q(a))$$

Proof

$$\dfrac{\dfrac{\dfrac{}{\forall\, a \bullet P(a) \land Q(a) \vdash \forall\, a \bullet P(a)}\;\texttt{allI} \qquad x \notin fv(\Gamma_1)\;\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{-}{Q(x) \vdash Q(x)}\;\texttt{asm}}{P(x), Q(x) \vdash Q(x)}\;\texttt{thin}}{P(x) \land Q(x) \vdash Q(x)}\;\texttt{conjE}}{\forall\, a \bullet P(a) \land Q(a) \vdash Q(x)}\;\texttt{allE}}{\forall\, a \bullet P(a) \land Q(a) \vdash \forall\, a \bullet Q(a)}\;\texttt{allI}}{\forall\, a \bullet P(a) \land Q(a) \vdash (\forall\, a \bullet P(a)) \land (\forall\, a \bullet Q(a))}\;\texttt{conjI}}{\vdash (\forall\, a \bullet P(a) \land Q(a)) \Rightarrow (\forall\, a \bullet P(a)) \land (\forall\, a \bullet Q(a))}\;\texttt{implI}$$

Example: Universal Quantification Distributes through Conjunction (L2R)

Natural Deduction Rules

$$(\forall\, a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall\, a \bullet P(a)) \wedge (\forall\, a \bullet Q(a))$$

Proof

$$\dfrac{\dfrac{\dfrac{}{\forall\, a \bullet P(a) \wedge Q(a) \vdash \forall\, a \bullet P(a)}\ \texttt{allI}\ \ x \notin fv(\Gamma_2) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{-}{Q(x) \vdash Q(x)}\ \texttt{asm}}{P(x), Q(x) \vdash Q(x)}\ \texttt{thin}}{P(x) \wedge Q(x) \vdash Q(x)}\ \texttt{conjE}}{\forall\, a \bullet P(a) \wedge Q(a) \vdash Q(x)}\ \texttt{allE}}{\forall\, a \bullet P(a) \wedge Q(a) \vdash \forall\, a \bullet Q(a)}\ \texttt{allI}\ \ x \notin fv(\Gamma_1)}{\forall\, a \bullet P(a) \wedge Q(a) \vdash (\forall\, a \bullet P(a)) \wedge (\forall\, a \bullet Q(a))}\ \texttt{conjI}}{\vdash (\forall\, a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall\, a \bullet P(a)) \wedge (\forall\, a \bullet Q(a))}\ \texttt{implI}$$

Example: Universal Quantification Distributes through Conjunction (L2R)

$$(\forall\, a \bullet P(a) \land Q(a)) \Rightarrow (\forall\, a \bullet P(a)) \land (\forall\, a \bullet Q(a))$$

Proof

$$
\dfrac{
  \dfrac{\forall\, a \bullet P(a) \land Q(a) \vdash P(x)}{\forall\, a \bullet P(a) \land Q(a) \vdash \forall\, a \bullet P(a)}\,\mathtt{allI}\; x \notin fv(\Gamma_2)
  \qquad
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{
          \dfrac{-}{Q(x) \vdash Q(x)}\,\mathtt{asm}
        }{P(x), Q(x) \vdash Q(x)}\,\mathtt{thin}
      }{P(x) \land Q(x) \vdash Q(x)}\,\mathtt{conjE}
    }{\forall\, a \bullet P(a) \land Q(a) \vdash Q(x)}\,\mathtt{allE}
  }{\forall\, a \bullet P(a) \land Q(a) \vdash \forall\, a \bullet Q(a)}\,\mathtt{allI}\; x \notin fv(\Gamma_1)
}{
  \dfrac{\forall\, a \bullet P(a) \land Q(a) \vdash (\forall\, a \bullet P(a)) \land (\forall\, a \bullet Q(a))}{\vdash (\forall\, a \bullet P(a) \land Q(a)) \Rightarrow (\forall\, a \bullet P(a)) \land (\forall\, a \bullet Q(a))}\,\mathtt{implI}
}\,\mathtt{conjI}
$$

Example: Universal Quantification Distributes through Conjunction (L2R)

Natural Deduction Rules

$$(\forall\, a \bullet P(a) \land Q(a)) \Rightarrow (\forall\, a \bullet P(a)) \land (\forall\, a \bullet Q(a))$$

Proof

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$\forall\, a \bullet P(a) \land Q(a) \vdash P(x)$}
\LeftLabel{$x \notin fv(\Gamma_2)$}
\RightLabel{$\mathtt{allI}$}
\UnaryInfC{$\forall\, a \bullet P(a) \land Q(a) \vdash \forall\, a \bullet P(a)$}
\AxiomC{$-$}
\RightLabel{$\mathtt{asm}$}
\UnaryInfC{$Q(x) \vdash Q(x)$}
\RightLabel{$\mathtt{thin}$}
\UnaryInfC{$P(x), Q(x) \vdash Q(x)$}
\RightLabel{$\mathtt{conjE}$}
\UnaryInfC{$P(x) \land Q(x) \vdash Q(x)$}
\RightLabel{$\mathtt{allE}$}
\UnaryInfC{$\forall\, a \bullet P(a) \land Q(a) \vdash Q(x)$}
\LeftLabel{$x \notin fv(\Gamma_1)$}
\RightLabel{$\mathtt{allI}$}
\UnaryInfC{$\forall\, a \bullet P(a) \land Q(a) \vdash \forall\, a \bullet Q(a)$}
\RightLabel{$\mathtt{conjI}$}
\BinaryInfC{$\forall\, a \bullet P(a) \land Q(a) \vdash (\forall\, a \bullet P(a)) \land (\forall\, a \bullet Q(a))$}
\RightLabel{$\mathtt{implI}$}
\UnaryInfC{$\vdash (\forall\, a \bullet P(a) \land Q(a)) \Rightarrow (\forall\, a \bullet P(a)) \land (\forall\, a \bullet Q(a))$}
\end{prooftree}
$$

Example: Universal Quantification Distributes through Conjunction (L2R)

Natural Deduction Rules

$$(\forall\, a \bullet P(a) \land Q(a)) \Rightarrow (\forall\, a \bullet P(a)) \land (\forall\, a \bullet Q(a))$$

Proof

$$\dfrac{\dfrac{x \notin fv(\Gamma_2)\quad \dfrac{\dfrac{}{\forall\, a \bullet P(a) \land Q(a) \vdash P(x)}\,\texttt{allE}}{\forall\, a \bullet P(a) \land Q(a) \vdash \forall\, a \bullet P(a)}\,\texttt{allI} \qquad x \notin fv(\Gamma_1)\quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{-}{Q(x) \vdash Q(x)}\,\texttt{asm}}{P(x), Q(x) \vdash Q(x)}\,\texttt{thin}}{P(x) \land Q(x) \vdash Q(x)}\,\texttt{conjE}}{\forall\, a \bullet P(a) \land Q(a) \vdash Q(x)}\,\texttt{allE}}{\forall\, a \bullet P(a) \land Q(a) \vdash \forall\, a \bullet Q(a)}\,\texttt{allI}}{\forall\, a \bullet P(a) \land Q(a) \vdash (\forall\, a \bullet P(a)) \land (\forall\, a \bullet Q(a))}\,\texttt{conjI}}{\vdash (\forall\, a \bullet P(a) \land Q(a)) \Rightarrow (\forall\, a \bullet P(a)) \land (\forall\, a \bullet Q(a))}\,\texttt{implI}$$

Example: Universal Quantification Distributes through Conjunction (L2R)

Natural Deduction Rules

$$(\forall a \bullet P(a) \land Q(a)) \Rightarrow (\forall a \bullet P(a)) \land (\forall a \bullet Q(a))$$

Proof

$$\dfrac{x \notin fv(\Gamma_2)\ \dfrac{\dfrac{P(x) \land Q(x) \vdash P(x)}{\forall a \bullet P(a) \land Q(a) \vdash P(x)}\,\texttt{allE}}{\forall a \bullet P(a) \land Q(a) \vdash \forall a \bullet P(a)}\,\texttt{allI} \qquad x \notin fv(\Gamma_1)\ \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{-}{Q(x) \vdash Q(x)}\,\texttt{asm}}{P(x), Q(x) \vdash Q(x)}\,\texttt{thin}}{P(x) \land Q(x) \vdash Q(x)}\,\texttt{conjE}}{\forall a \bullet P(a) \land Q(a) \vdash Q(x)}\,\texttt{allE}}{\forall a \bullet P(a) \land Q(a) \vdash \forall a \bullet Q(a)}\,\texttt{allI}}{\dfrac{\forall a \bullet P(a) \land Q(a) \vdash (\forall a \bullet P(a)) \land (\forall a \bullet Q(a))}{\vdash (\forall a \bullet P(a) \land Q(a)) \Rightarrow (\forall a \bullet P(a)) \land (\forall a \bullet Q(a))}\,\texttt{implI}}\,\texttt{conjI}$$

Example: Universal Quantification Distributes through Conjunction (L2R)

Natural Deduction Rules

$$(\forall\, a \bullet P(a) \land Q(a)) \Rightarrow (\forall\, a \bullet P(a)) \land (\forall\, a \bullet Q(a))$$

Proof

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$P(x) \land Q(x) \vdash P(x)$}
\RightLabel{\texttt{allE}}
\UnaryInfC{$\forall\, a \bullet P(a) \land Q(a) \vdash P(x)$}
\LeftLabel{$x \notin fv(\Gamma_2)$}
\RightLabel{\texttt{allI}}
\UnaryInfC{$\forall\, a \bullet P(a) \land Q(a) \vdash \forall\, a \bullet P(a)$}
\AxiomC{$-$}
\RightLabel{\texttt{asm}}
\UnaryInfC{$Q(x) \vdash Q(x)$}
\RightLabel{\texttt{thin}}
\UnaryInfC{$P(x), Q(x) \vdash Q(x)$}
\RightLabel{\texttt{conjE}}
\UnaryInfC{$P(x) \land Q(x) \vdash Q(x)$}
\RightLabel{\texttt{allE}}
\UnaryInfC{$\forall\, a \bullet P(a) \land Q(a) \vdash Q(x)$}
\LeftLabel{$x \notin fv(\Gamma_1)$}
\RightLabel{\texttt{allI}}
\UnaryInfC{$\forall\, a \bullet P(a) \land Q(a) \vdash \forall\, a \bullet Q(a)$}
\RightLabel{\texttt{conjI}}
\BinaryInfC{$\forall\, a \bullet P(a) \land Q(a) \vdash (\forall\, a \bullet P(a)) \land (\forall\, a \bullet Q(a))$}
\RightLabel{\texttt{implI}}
\UnaryInfC{$\vdash (\forall\, a \bullet P(a) \land Q(a)) \Rightarrow (\forall\, a \bullet P(a)) \land (\forall\, a \bullet Q(a))$}
\end{prooftree}
$$

Example: Universal Quantification Distributes through Conjunction (L2R)

Natural Deduction Rules

$$(\forall\, a \bullet P(a) \land Q(a)) \Rightarrow (\forall\, a \bullet P(a)) \land (\forall\, a \bullet Q(a))$$

Proof

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{$\texttt{conjE}$}
\UnaryInfC{$P(x) \land Q(x) \vdash P(x)$}
\RightLabel{$\texttt{allE}$}
\UnaryInfC{$\forall\, a \bullet P(a) \land Q(a) \vdash P(x)$}
\LeftLabel{$x \notin fv(\Gamma_2)$}
\RightLabel{$\texttt{allI}$}
\UnaryInfC{$\forall\, a \bullet P(a) \land Q(a) \vdash \forall\, a \bullet P(a)$}
\AxiomC{$-$}
\RightLabel{$\texttt{asm}$}
\UnaryInfC{$Q(x) \vdash Q(x)$}
\RightLabel{$\texttt{thin}$}
\UnaryInfC{$P(x), Q(x) \vdash Q(x)$}
\RightLabel{$\texttt{conjE}$}
\UnaryInfC{$P(x) \land Q(x) \vdash Q(x)$}
\RightLabel{$\texttt{allE}$}
\UnaryInfC{$\forall\, a \bullet P(a) \land Q(a) \vdash Q(x)$}
\LeftLabel{$x \notin fv(\Gamma_1)$}
\RightLabel{$\texttt{allI}$}
\UnaryInfC{$\forall\, a \bullet P(a) \land Q(a) \vdash \forall\, a \bullet Q(a)$}
\RightLabel{$\texttt{conjI}$}
\BinaryInfC{$\forall\, a \bullet P(a) \land Q(a) \vdash (\forall\, a \bullet P(a)) \land (\forall\, a \bullet Q(a))$}
\RightLabel{$\texttt{implI}$}
\UnaryInfC{$\vdash (\forall\, a \bullet P(a) \land Q(a)) \Rightarrow (\forall\, a \bullet P(a)) \land (\forall\, a \bullet Q(a))$}
\end{prooftree}
$$

Example: Universal Quantification Distributes through Conjunction (L2R)

$$(\forall a \bullet P(a) \land Q(a)) \Rightarrow (\forall a \bullet P(a)) \land (\forall a \bullet Q(a))$$

Proof

$$
\begin{prooftree}
\AxiomC{$P(x), Q(x) \vdash P(x)$}
\RightLabel{\texttt{conjE}}
\UnaryInfC{$P(x) \land Q(x) \vdash P(x)$}
\RightLabel{\texttt{allE}}
\UnaryInfC{$\forall a \bullet P(a) \land Q(a) \vdash P(x)$}
\LeftLabel{$x \notin fv(\Gamma_2)$}
\RightLabel{\texttt{allI}}
\UnaryInfC{$\forall a \bullet P(a) \land Q(a) \vdash \forall a \bullet P(a)$}
\AxiomC{$-$}
\RightLabel{\texttt{asm}}
\UnaryInfC{$Q(x) \vdash Q(x)$}
\RightLabel{\texttt{thin}}
\UnaryInfC{$P(x), Q(x) \vdash Q(x)$}
\RightLabel{\texttt{conjE}}
\UnaryInfC{$P(x) \land Q(x) \vdash Q(x)$}
\RightLabel{\texttt{allE}}
\UnaryInfC{$\forall a \bullet P(a) \land Q(a) \vdash Q(x)$}
\LeftLabel{$x \notin fv(\Gamma_1)$}
\RightLabel{\texttt{allI}}
\UnaryInfC{$\forall a \bullet P(a) \land Q(a) \vdash \forall a \bullet Q(a)$}
\RightLabel{\texttt{conjI}}
\BinaryInfC{$\forall a \bullet P(a) \land Q(a) \vdash (\forall a \bullet P(a)) \land (\forall a \bullet Q(a))$}
\RightLabel{\texttt{implI}}
\UnaryInfC{$\vdash (\forall a \bullet P(a) \land Q(a)) \Rightarrow (\forall a \bullet P(a)) \land (\forall a \bullet Q(a))$}
\end{prooftree}
$$

Example: Universal Quantification Distributes through Conjunction (L2R)

Natural Deduction Rules

$$(\forall a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))$$

Proof

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$P(x), Q(x) \vdash P(x)$}
\RightLabel{\texttt{conjE}}
\UnaryInfC{$P(x) \wedge Q(x) \vdash P(x)$}
\RightLabel{\texttt{allE}}
\UnaryInfC{$\forall a \bullet P(a) \wedge Q(a) \vdash P(x)$}
\LeftLabel{$x \notin fv(\Gamma_2)$}
\RightLabel{\texttt{allI}}
\UnaryInfC{$\forall a \bullet P(a) \wedge Q(a) \vdash \forall a \bullet P(a)$}
\AxiomC{$-$}
\RightLabel{\texttt{asm}}
\UnaryInfC{$Q(x) \vdash Q(x)$}
\RightLabel{\texttt{thin}}
\UnaryInfC{$P(x), Q(x) \vdash Q(x)$}
\RightLabel{\texttt{conjE}}
\UnaryInfC{$P(x) \wedge Q(x) \vdash Q(x)$}
\RightLabel{\texttt{allE}}
\UnaryInfC{$\forall a \bullet P(a) \wedge Q(a) \vdash Q(x)$}
\LeftLabel{$x \notin fv(\Gamma_1)$}
\RightLabel{\texttt{allI}}
\UnaryInfC{$\forall a \bullet P(a) \wedge Q(a) \vdash \forall a \bullet Q(a)$}
\RightLabel{\texttt{conjI}}
\BinaryInfC{$\forall a \bullet P(a) \wedge Q(a) \vdash (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))$}
\RightLabel{\texttt{implI}}
\UnaryInfC{$\vdash (\forall a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))$}
\end{prooftree}
$$

Example: Universal Quantification Distributes through Conjunction (L2R)

Natural Deduction Rules

$$(\forall\, a \bullet P(a) \land Q(a)) \Rightarrow (\forall\, a \bullet P(a)) \land (\forall\, a \bullet Q(a))$$

Proof

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{
          \dfrac{}{P(x), Q(x) \vdash P(x)}\ \mathtt{asm}
        }{P(x) \land Q(x) \vdash P(x)}\ \mathtt{conjE}
      }{\forall\, a \bullet P(a) \land Q(a) \vdash P(x)}\ \mathtt{allE}
    }{\forall\, a \bullet P(a) \land Q(a) \vdash \forall\, a \bullet P(a)}\ \mathtt{allI}\ \ x \notin fv(\Gamma_2)
    \qquad
    \dfrac{
      \dfrac{
        \dfrac{
          \dfrac{
            \dfrac{-}{Q(x) \vdash Q(x)}\ \mathtt{asm}
          }{P(x), Q(x) \vdash Q(x)}\ \mathtt{thin}
        }{P(x) \land Q(x) \vdash Q(x)}\ \mathtt{conjE}
      }{\forall\, a \bullet P(a) \land Q(a) \vdash Q(x)}\ \mathtt{allE}
    }{\forall\, a \bullet P(a) \land Q(a) \vdash \forall\, a \bullet Q(a)}\ \mathtt{allI}\ \ x \notin fv(\Gamma_1)
  }{\forall\, a \bullet P(a) \land Q(a) \vdash (\forall\, a \bullet P(a)) \land (\forall\, a \bullet Q(a))}\ \mathtt{conjI}
}{\vdash (\forall\, a \bullet P(a) \land Q(a)) \Rightarrow (\forall\, a \bullet P(a)) \land (\forall\, a \bullet Q(a))}\ \mathtt{implI}
$$

Example: Universal Quantification Distributes through Conjunction (L2R)

$$(\forall\, a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall\, a \bullet P(a)) \wedge (\forall\, a \bullet Q(a))$$

Proof

$$
\begin{prooftree}
\AxiomC{$-$}
\RightLabel{$\mathtt{asm}$}
\UnaryInfC{$P(x), Q(x) \vdash P(x)$}
\RightLabel{$\mathtt{conjE}$}
\UnaryInfC{$P(x) \wedge Q(x) \vdash P(x)$}
\RightLabel{$\mathtt{allE}$}
\UnaryInfC{$\forall\, a \bullet P(a) \wedge Q(a) \vdash P(x)$}
\LeftLabel{$x \notin fv(\Gamma_2)$}
\RightLabel{$\mathtt{allI}$}
\UnaryInfC{$\forall\, a \bullet P(a) \wedge Q(a) \vdash \forall\, a \bullet P(a)$}
\AxiomC{$-$}
\RightLabel{$\mathtt{asm}$}
\UnaryInfC{$Q(x) \vdash Q(x)$}
\RightLabel{$\mathtt{thin}$}
\UnaryInfC{$P(x), Q(x) \vdash Q(x)$}
\RightLabel{$\mathtt{conjE}$}
\UnaryInfC{$P(x) \wedge Q(x) \vdash Q(x)$}
\RightLabel{$\mathtt{allE}$}
\UnaryInfC{$\forall\, a \bullet P(a) \wedge Q(a) \vdash Q(x)$}
\LeftLabel{$x \notin fv(\Gamma_1)$}
\RightLabel{$\mathtt{allI}$}
\UnaryInfC{$\forall\, a \bullet P(a) \wedge Q(a) \vdash \forall\, a \bullet Q(a)$}
\RightLabel{$\mathtt{conjI}$}
\BinaryInfC{$\forall\, a \bullet P(a) \wedge Q(a) \vdash (\forall\, a \bullet P(a)) \wedge (\forall\, a \bullet Q(a))$}
\RightLabel{$\mathtt{implI}$}
\UnaryInfC{$\vdash (\forall\, a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall\, a \bullet P(a)) \wedge (\forall\, a \bullet Q(a))$}
\end{prooftree}
$$

Example: Universal Quantification Distributes through Conjunction (L2R)

Natural Deduction Rules

$$(\forall\, a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall\, a \bullet P(a)) \wedge (\forall\, a \bullet Q(a))$$

Proof

$$
\begin{prooftree}
\AxiomC{$-$}
\RightLabel{$\mathtt{asm}$}
\UnaryInfC{$P(x), Q(x) \vdash P(x)$}
\RightLabel{$\mathtt{conjE}$}
\UnaryInfC{$P(x) \wedge Q(x) \vdash P(x)$}
\RightLabel{$\mathtt{allE}$}
\UnaryInfC{$\forall\, a \bullet P(a) \wedge Q(a) \vdash P(x)$}
\LeftLabel{$x \notin fv(\Gamma_2)$}
\RightLabel{$\mathtt{allI}$}
\UnaryInfC{$\forall\, a \bullet P(a) \wedge Q(a) \vdash \forall\, a \bullet P(a)$}
\AxiomC{$-$}
\RightLabel{$\mathtt{asm}$}
\UnaryInfC{$Q(x) \vdash Q(x)$}
\RightLabel{$\mathtt{thin}$}
\UnaryInfC{$P(x), Q(x) \vdash Q(x)$}
\RightLabel{$\mathtt{conjE}$}
\UnaryInfC{$P(x) \wedge Q(x) \vdash Q(x)$}
\RightLabel{$\mathtt{allE}$}
\UnaryInfC{$\forall\, a \bullet P(a) \wedge Q(a) \vdash Q(x)$}
\LeftLabel{$x \notin fv(\Gamma_1)$}
\RightLabel{$\mathtt{allI}$}
\UnaryInfC{$\forall\, a \bullet P(a) \wedge Q(a) \vdash \forall\, a \bullet Q(a)$}
\RightLabel{$\mathtt{conjI}$}
\BinaryInfC{$\forall\, a \bullet P(a) \wedge Q(a) \vdash (\forall\, a \bullet P(a)) \wedge (\forall\, a \bullet Q(a))$}
\RightLabel{$\mathtt{implI}$}
\UnaryInfC{$\vdash (\forall\, a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall\, a \bullet P(a)) \wedge (\forall\, a \bullet Q(a))$}
\end{prooftree}
$$

$x \notin fv(\Gamma_1)$

Example: Universal Quantification Distributes through Conjunction (L2R)

Natural Deduction Rules

$$(\forall\, a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall\, a \bullet P(a)) \wedge (\forall\, a \bullet Q(a))$$

Proof

$$\dfrac{\dfrac{\dfrac{x \notin fv(\Gamma_2)\;\dfrac{\dfrac{\dfrac{\dfrac{-}{P(x), Q(x) \vdash P(x)}\;\mathtt{asm}}{P(x) \wedge Q(x) \vdash P(x)}\;\mathtt{conjE}}{\forall\, a \bullet P(a) \wedge Q(a) \vdash P(x)}\;\mathtt{allE}}{\forall\, a \bullet P(a) \wedge Q(a) \vdash \forall\, a \bullet P(a)}\;\mathtt{allI} \qquad x \notin fv(\Gamma_1)\;\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{-}{Q(x) \vdash Q(x)}\;\mathtt{asm}}{P(x), Q(x) \vdash Q(x)}\;\mathtt{thin}}{P(x) \wedge Q(x) \vdash Q(x)}\;\mathtt{conjE}}{\forall\, a \bullet P(a) \wedge Q(a) \vdash Q(x)}\;\mathtt{allE}}{\forall\, a \bullet P(a) \wedge Q(a) \vdash \forall\, a \bullet Q(a)}\;\mathtt{allI}}{\forall\, a \bullet P(a) \wedge Q(a) \vdash (\forall\, a \bullet P(a)) \wedge (\forall\, a \bullet Q(a))}\;\mathtt{conjI}}{\vdash (\forall\, a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall\, a \bullet P(a)) \wedge (\forall\, a \bullet Q(a))}\;\mathtt{implI}}{}$$

$$x \notin fv(\Gamma_1) \;=\; x \notin fv(\Gamma_2)$$

Example: Universal Quantification Distributes through Conjunction (L2R)

Natural Deduction Rules

$$(\forall\, a \bullet P(a) \land Q(a)) \Rightarrow (\forall\, a \bullet P(a)) \land (\forall\, a \bullet Q(a))$$

Proof

$$
\dfrac{
  \dfrac{
    x \notin fv(\Gamma_2)\;
    \dfrac{
      \dfrac{
        \dfrac{
          \dfrac{-}{P(x), Q(x) \vdash P(x)}\,\texttt{asm}
        }{P(x) \land Q(x) \vdash P(x)}\,\texttt{conjE}
      }{\forall\, a \bullet P(a) \land Q(a) \vdash P(x)}\,\texttt{allE}
    }{\forall\, a \bullet P(a) \land Q(a) \vdash \forall\, a \bullet P(a)}\,\texttt{allI}
    \quad
    x \notin fv(\Gamma_1)\;
    \dfrac{
      \dfrac{
        \dfrac{
          \dfrac{
            \dfrac{-}{Q(x) \vdash Q(x)}\,\texttt{asm}
          }{P(x), Q(x) \vdash Q(x)}\,\texttt{thin}
        }{P(x) \land Q(x) \vdash Q(x)}\,\texttt{conjE}
      }{\forall\, a \bullet P(a) \land Q(a) \vdash Q(x)}\,\texttt{allE}
    }{\forall\, a \bullet P(a) \land Q(a) \vdash \forall\, a \bullet Q(a)}\,\texttt{allI}
  }{\forall\, a \bullet P(a) \land Q(a) \vdash (\forall\, a \bullet P(a)) \land (\forall\, a \bullet Q(a))}\,\texttt{conjI}
}{\vdash (\forall\, a \bullet P(a) \land Q(a)) \Rightarrow (\forall\, a \bullet P(a)) \land (\forall\, a \bullet Q(a))}\,\texttt{implI}
$$

$$x \notin fv(\Gamma_1) \;=\; x \notin fv(\Gamma_2) \;=\; x \notin fv(\forall\, a \bullet P(a) \land Q(a))$$

Example: Universal Quantification Distributes through Conjunction (L2R)

Natural Deduction Rules

$$(\forall\, a \bullet P(a) \land Q(a)) \Rightarrow (\forall\, a \bullet P(a)) \land (\forall\, a \bullet Q(a))$$

Proof

$$\dfrac{x \notin fv(\Gamma_2)\ \dfrac{\dfrac{\dfrac{\dfrac{-}{P(x), Q(x) \vdash P(x)}\,\texttt{asm}}{P(x) \land Q(x) \vdash P(x)}\,\texttt{conjE}}{\forall\, a \bullet P(a) \land Q(a) \vdash P(x)}\,\texttt{allE}}{\forall\, a \bullet P(a) \land Q(a) \vdash \forall\, a \bullet P(a)}\,\texttt{allI} \qquad x \notin fv(\Gamma_1)\ \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{-}{Q(x) \vdash Q(x)}\,\texttt{asm}}{P(x), Q(x) \vdash Q(x)}\,\texttt{thin}}{P(x) \land Q(x) \vdash Q(x)}\,\texttt{conjE}}{\forall\, a \bullet P(a) \land Q(a) \vdash Q(x)}\,\texttt{allE}}{\forall\, a \bullet P(a) \land Q(a) \vdash \forall\, a \bullet Q(a)}\,\texttt{allI}}{\dfrac{\forall\, a \bullet P(a) \land Q(a) \vdash (\forall\, a \bullet P(a)) \land (\forall\, a \bullet Q(a))}{\vdash (\forall\, a \bullet P(a) \land Q(a)) \Rightarrow (\forall\, a \bullet P(a)) \land (\forall\, a \bullet Q(a))}\,\texttt{implI}}\,\texttt{conjI}$$

$$x \notin fv(\Gamma_1) \;=\; x \notin fv(\Gamma_2) \;=\; x \notin fv(\forall\, a \bullet P(a) \land Q(a)) \;=\; \mathit{true}$$

Outline

Natural Deduction Rules

Universal Quantification

Existential Quantification

Existential Quantification

- How do we reason with the existential quantifier?
- It's a generalisation of disjunction, so take inspiration from the `disjI` rules

$$\frac{\Gamma \vdash P}{\Gamma \vdash P \vee Q}\ \texttt{disjI1}$$

$$\frac{\dfrac{\vdots}{\vdash P(k)}}{\vdash \exists i : \mathbb{N} \bullet P(i)}\ \texttt{disjIk}$$

- Here we choose k out of the infinite possibilities!

Existential Quantification

- How do we reason with the existential quantifier?
- It's a generalisation of disjunction, so take inspiration from the `disjI` rules

$$\frac{\Gamma \vdash P}{\Gamma \vdash P \lor Q}\ \texttt{disjI1}$$

$$\frac{\dfrac{\vdots}{\vdash P(k)}}{\vdash \exists i : \mathbb{N} \bullet P(i)}\ \texttt{disjIk}$$

- Here we choose k out of the infinite possibilities!

Existential Quantification

- How do we reason with the existential quantifier?
- It's a generalisation of disjunction, so take inspiration from the `disjI` rules

$$\frac{\Gamma \vdash P}{\Gamma \vdash P \lor Q}\ \texttt{disjI1}$$

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- Here we choose k out of the infinite possibilities!

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- Example: $\exists\, a \bullet a \geq a^2$ is true since $a = 0$ is a solution. There are many others.
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- Existential elimination is the dual of universal introduction.
- Existential elimination: $x \notin fv(\Gamma, Q)\ \dfrac{P(x), \Gamma \vdash Q}{\exists a \bullet P(a), \Gamma \vdash Q}\ \mathtt{exE}$
- Like universal introduction, there is an important side condition.
- The fixed variable must be arbitrary: $x \notin fv(\Gamma, Q)$.
- Existential elimination allows us to derive an arbitrary instance of a predicate.
- We can then derive conclusions from that arbitrary instance.
- Remember: it's a potentially infinite case analysis.
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Existential Elimination

- Existential elimination is the dual of universal introduction.
- Existential elimination: $x \notin fv(\Gamma, Q) \dfrac{P(x), \Gamma \vdash Q}{\exists\, a \bullet P(a), \Gamma \vdash Q}\,\texttt{exE}$
- Like universal introduction, there is an important side condition.
- The fixed variable must be arbitrary: $x \notin fv(\Gamma, Q)$.
- Existential elimination allows us to derive an arbitrary instance of a predicate.
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Example: Existential Quantifier and Bound Variable

- Convention

  $\exists\, a \bullet P(a)$ versus $\exists\, a \bullet P$

  $P(a)$ means a is free for substitution

  P means a is not free for substitution

- Example $(\exists\, a \bullet P) \Leftrightarrow P$

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$$

Example: Reasoning with Existential Quantification

$$(\exists\, a \bullet P(a) \land Q(a)) \Rightarrow (\exists\, a \bullet P(a))$$

$$\dfrac{x \notin fv(\exists\, a \bullet P(a)) \qquad \dfrac{\dfrac{\dfrac{\dfrac{-}{P(x), Q(x) \vdash P(x)}\ \texttt{asm}}{P(x), Q(x) \vdash \exists\, a \bullet P(a)}\ \texttt{exI}}{P(x) \land Q(x) \vdash \exists\, a \bullet P(a)}\ \texttt{conjE}}{\exists\, a \bullet P(a) \land Q(a) \vdash \exists\, a \bullet P(a)}\ \texttt{exE}}{\vdash (\exists\, a \bullet P(a) \land Q(a)) \Rightarrow (\exists\, a \bullet P(a))}\ \texttt{impI}$$

Example: Existential Quantifiers Commute

$$(\exists\, a \bullet \exists\, b \bullet P(a, b)) \Rightarrow (\exists\, b \bullet \exists\, a \bullet P(a, b))$$

Example: Existential Quantifiers Commute

$$(\exists\, a \bullet \exists\, b \bullet P(a, b)) \Rightarrow (\exists\, b \bullet \exists\, a \bullet P(a, b))$$

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$$(\exists\, a \bullet \exists\, b \bullet P(a, b)) \Rightarrow (\exists\, b \bullet \exists\, a \bullet P(a, b))$$

$$\vdash (\exists\, a \bullet \exists\, b \bullet P(a, b)) \Rightarrow (\exists\, b \bullet \exists\, a \bullet P(a, b))$$

Example: Existential Quantifiers Commute

$$(\exists\, a \bullet \exists\, b \bullet P(a,b)) \Rightarrow (\exists\, b \bullet \exists\, a \bullet P(a,b))$$

$$\frac{}{\vdash (\exists\, a \bullet \exists\, b \bullet P(a,b)) \Rightarrow (\exists\, b \bullet \exists\, a \bullet P(a,b))}$$

Example: Existential Quantifiers Commute

$$(\exists\, a \bullet \exists\, b \bullet P(a,b)) \Rightarrow (\exists\, b \bullet \exists\, a \bullet P(a,b))$$

$$\dfrac{}{\vdash (\exists\, a \bullet \exists\, b \bullet P(a,b)) \Rightarrow (\exists\, b \bullet \exists\, a \bullet P(a,b))}\ \texttt{impI}$$

Example: Existential Quantifiers Commute

$$(\exists\, a \bullet \exists\, b \bullet P(a, b)) \Rightarrow (\exists\, b \bullet \exists\, a \bullet P(a, b))$$

$$\dfrac{\exists\, a \bullet \exists\, b \bullet P(a, b) \vdash \exists\, b \bullet \exists\, a \bullet P(a, b)}{\vdash (\exists\, a \bullet \exists\, b \bullet P(a, b)) \Rightarrow (\exists\, b \bullet \exists\, a \bullet P(a, b))}\ \texttt{impI}$$

Example: Existential Quantifiers Commute

$$(\exists\, a \bullet \exists\, b \bullet P(a, b)) \Rightarrow (\exists\, b \bullet \exists\, a \bullet P(a, b))$$

$$\dfrac{\dfrac{}{\exists\, a \bullet \exists\, b \bullet P(a, b) \vdash \exists\, b \bullet \exists\, a \bullet P(a, b)}}{\vdash (\exists\, a \bullet \exists\, b \bullet P(a, b)) \Rightarrow (\exists\, b \bullet \exists\, a \bullet P(a, b))}\ \texttt{impI}$$

Example: Existential Quantifiers Commute

$$(\exists\, a \bullet \exists\, b \bullet P(a, b)) \Rightarrow (\exists\, b \bullet \exists\, a \bullet P(a, b))$$

$$\dfrac{\dfrac{}{\exists\, a \bullet \exists\, b \bullet P(a, b) \vdash \exists\, b \bullet \exists\, a \bullet P(a, b)}\ \texttt{exE}}{\vdash (\exists\, a \bullet \exists\, b \bullet P(a, b)) \Rightarrow (\exists\, b \bullet \exists\, a \bullet P(a, b))}\ \texttt{impI}$$

Example: Existential Quantifiers Commute

$$(\exists\, a \bullet \exists\, b \bullet P(a,b)) \Rightarrow (\exists\, b \bullet \exists\, a \bullet P(a,b))$$

$$x \notin fv(\exists\, b \bullet \exists\, a \bullet P(a,b)) \quad \dfrac{\dfrac{}{\exists\, a \bullet \exists\, b \bullet P(a,b) \vdash \exists\, b \bullet \exists\, a \bullet P(a,b)}\ \texttt{exE}}{\vdash (\exists\, a \bullet \exists\, b \bullet P(a,b)) \Rightarrow (\exists\, b \bullet \exists\, a \bullet P(a,b))}\ \texttt{impI}$$

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$$(\exists\, a \bullet \exists\, b \bullet P(a, b)) \Rightarrow (\exists\, b \bullet \exists\, a \bullet P(a, b))$$

$$\dfrac{x \notin fv(\exists\, b \bullet \exists\, a \bullet P(a, b)) \quad \dfrac{\exists\, b \bullet P(x, b) \vdash \exists\, b \bullet \exists\, a \bullet P(a, b)}{\exists\, a \bullet \exists\, b \bullet P(a, b) \vdash \exists\, b \bullet \exists\, a \bullet P(a, b)}\ \mathtt{exE}}{\vdash (\exists\, a \bullet \exists\, b \bullet P(a, b)) \Rightarrow (\exists\, b \bullet \exists\, a \bullet P(a, b))}\ \mathtt{impI}$$

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$$(\exists\, a \bullet \exists\, b \bullet P(a, b)) \Rightarrow (\exists\, b \bullet \exists\, a \bullet P(a, b))$$

$$\dfrac{x \notin fv(\exists\, b \bullet \exists\, a \bullet P(a, b)) \quad \dfrac{}{\exists\, b \bullet P(x, b) \vdash \exists\, b \bullet \exists\, a \bullet P(a, b)}}{\dfrac{\exists\, a \bullet \exists\, b \bullet P(a, b) \vdash \exists\, b \bullet \exists\, a \bullet P(a, b)}{\vdash (\exists\, a \bullet \exists\, b \bullet P(a, b)) \Rightarrow (\exists\, b \bullet \exists\, a \bullet P(a, b))}\ \mathtt{impI}}\ \mathtt{exE}$$

Example: Existential Quantifiers Commute

$$(\exists\, a \bullet \exists\, b \bullet P(a, b)) \Rightarrow (\exists\, b \bullet \exists\, a \bullet P(a, b))$$

$$\dfrac{x \notin fv(\exists\, b \bullet \exists\, a \bullet P(a, b)) \quad \dfrac{\dfrac{}{\exists\, b \bullet P(x, b) \vdash \exists\, b \bullet \exists\, a \bullet P(a, b)}\,\mathtt{exE}}{\exists\, a \bullet \exists\, b \bullet P(a, b) \vdash \exists\, b \bullet \exists\, a \bullet P(a, b)}\,\mathtt{exE}}{\vdash (\exists\, a \bullet \exists\, b \bullet P(a, b)) \Rightarrow (\exists\, b \bullet \exists\, a \bullet P(a, b))}\,\mathtt{impI}$$

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$$(\exists\, a \bullet \exists\, b \bullet P(a,b)) \Rightarrow (\exists\, b \bullet \exists\, a \bullet P(a,b))$$

$$\dfrac{x \notin fv(\exists\, b \bullet \exists\, a \bullet P(a,b)) \quad \dfrac{y \notin fv(\exists\, b \bullet \exists\, a \bullet P(a,b)) \quad \dfrac{P(x,y) \vdash \exists\, b \bullet \exists\, a \bullet P(a,b)}{\exists\, b \bullet P(x,b) \vdash \exists\, b \bullet \exists\, a \bullet P(a,b)}\,\mathtt{exE}}{\exists\, a \bullet \exists\, b \bullet P(a,b) \vdash \exists\, b \bullet \exists\, a \bullet P(a,b)}\,\mathtt{exE}}{\vdash (\exists\, a \bullet \exists\, b \bullet P(a,b)) \Rightarrow (\exists\, b \bullet \exists\, a \bullet P(a,b))}\,\mathtt{impI}$$

Example: Existential Quantifiers Commute

$$(\exists\, a \bullet \exists\, b \bullet P(a, b)) \Rightarrow (\exists\, b \bullet \exists\, a \bullet P(a, b))$$

$$\dfrac{x \notin fv(\exists\, b \bullet \exists\, a \bullet P(a, b)) \quad \dfrac{y \notin fv(\exists\, b \bullet \exists\, a \bullet P(a, b)) \quad \dfrac{\dfrac{}{P(x, y) \vdash \exists\, b \bullet \exists\, a \bullet P(a, b)}}{\exists\, b \bullet P(x, b) \vdash \exists\, b \bullet \exists\, a \bullet P(a, b)}\ \mathtt{exE}}{\exists\, a \bullet \exists\, b \bullet P(a, b) \vdash \exists\, b \bullet \exists\, a \bullet P(a, b)}\ \mathtt{exE}}{\vdash (\exists\, a \bullet \exists\, b \bullet P(a, b)) \Rightarrow (\exists\, b \bullet \exists\, a \bullet P(a, b))}\ \mathtt{impI}$$

Example: Existential Quantifiers Commute

$$(\exists\, a \bullet \exists\, b \bullet P(a,b)) \Rightarrow (\exists\, b \bullet \exists\, a \bullet P(a,b))$$

$$\dfrac{x \notin fv(\exists\, b \bullet \exists\, a \bullet P(a,b)) \quad \dfrac{y \notin fv(\exists\, b \bullet \exists\, a \bullet P(a,b)) \quad \dfrac{\dfrac{}{P(x,y) \vdash \exists\, b \bullet \exists\, a \bullet P(a,b)}\ \mathtt{exI}}{\exists\, b \bullet P(x,b) \vdash \exists\, b \bullet \exists\, a \bullet P(a,b)}\ \mathtt{exE}}{\exists\, a \bullet \exists\, b \bullet P(a,b) \vdash \exists\, b \bullet \exists\, a \bullet P(a,b)}\ \mathtt{exE}}{\vdash (\exists\, a \bullet \exists\, b \bullet P(a,b)) \Rightarrow (\exists\, b \bullet \exists\, a \bullet P(a,b))}\ \mathtt{impI}$$

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$$\dfrac{x \notin fv(\exists\, b \bullet \exists\, a \bullet P(a,b)) \quad \dfrac{y \notin fv(\exists\, b \bullet \exists\, a \bullet P(a,b)) \quad \dfrac{\dfrac{P(x,y) \vdash \exists\, a \bullet P(a,y)}{P(x,y) \vdash \exists\, b \bullet \exists\, a \bullet P(a,b)}\ \texttt{exI}}{\exists\, b \bullet P(x,b) \vdash \exists\, b \bullet \exists\, a \bullet P(a,b)}\ \texttt{exE}}{\exists\, a \bullet \exists\, b \bullet P(a,b) \vdash \exists\, b \bullet \exists\, a \bullet P(a,b)}\ \texttt{exE}}{\vdash (\exists\, a \bullet \exists\, b \bullet P(a,b)) \Rightarrow (\exists\, b \bullet \exists\, a \bullet P(a,b))}\ \texttt{impI}$$

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$$(\exists\, a \bullet \exists\, b \bullet P(a,b)) \Rightarrow (\exists\, b \bullet \exists\, a \bullet P(a,b))$$

$$\cfrac{x \notin fv(\exists\, b \bullet \exists\, a \bullet P(a,b)) \quad \cfrac{y \notin fv(\exists\, b \bullet \exists\, a \bullet P(a,b)) \quad \cfrac{\cfrac{}{P(x,y) \vdash \exists\, a \bullet P(a,y)}}{P(x,y) \vdash \exists\, b \bullet \exists\, a \bullet P(a,b)}\ \mathtt{exI}}{\exists\, b \bullet P(x,b) \vdash \exists\, b \bullet \exists\, a \bullet P(a,b)}\ \mathtt{exE}}{\cfrac{\exists\, a \bullet \exists\, b \bullet P(a,b) \vdash \exists\, b \bullet \exists\, a \bullet P(a,b)}{\vdash (\exists\, a \bullet \exists\, b \bullet P(a,b)) \Rightarrow (\exists\, b \bullet \exists\, a \bullet P(a,b))}\ \mathtt{impI}}\ \mathtt{exE}$$

Example: Existential Quantifiers Commute

$$(\exists\, a \bullet \exists\, b \bullet P(a, b)) \Rightarrow (\exists\, b \bullet \exists\, a \bullet P(a, b))$$

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$$(\exists\, a \bullet \exists\, b \bullet P(a,b)) \Rightarrow (\exists\, b \bullet \exists\, a \bullet P(a,b))$$

$$
\dfrac{x \notin fv(\exists\, b \bullet \exists\, a \bullet P(a,b)) \quad \dfrac{y \notin fv(\exists\, b \bullet \exists\, a \bullet P(a,b)) \quad \dfrac{\dfrac{P(x,y) \vdash P(x,y)}{P(x,y) \vdash \exists\, a \bullet P(a,y)}\;\mathtt{exI}}{P(x,y) \vdash \exists\, b \bullet \exists\, a \bullet P(a,b)}\;\mathtt{exI}}{\exists\, b \bullet P(x,b) \vdash \exists\, b \bullet \exists\, a \bullet P(a,b)}\;\mathtt{exE}}{\dfrac{\exists\, a \bullet \exists\, b \bullet P(a,b) \vdash \exists\, b \bullet \exists\, a \bullet P(a,b)}{\vdash (\exists\, a \bullet \exists\, b \bullet P(a,b)) \Rightarrow (\exists\, b \bullet \exists\, a \bullet P(a,b))}\;\mathtt{impI}}\;\mathtt{exE}
$$

Example: Existential Quantifiers Commute

$$(\exists\, a \bullet \exists\, b \bullet P(a, b)) \Rightarrow (\exists\, b \bullet \exists\, a \bullet P(a, b))$$

$$\dfrac{x \notin fv(\exists\, b \bullet \exists\, a \bullet P(a,b)) \quad \dfrac{y \notin fv(\exists\, b \bullet \exists\, a \bullet P(a,b)) \quad \dfrac{\dfrac{\dfrac{}{P(x,y) \vdash P(x,y)}}{P(x,y) \vdash \exists\, a \bullet P(a,y)}\;\texttt{exI}}{P(x,y) \vdash \exists\, b \bullet \exists\, a \bullet P(a,b)}\;\texttt{exI}}{\exists\, b \bullet P(x,b) \vdash \exists\, b \bullet \exists\, a \bullet P(a,b)}\;\texttt{exE}}{\dfrac{\exists\, a \bullet \exists\, b \bullet P(a,b) \vdash \exists\, b \bullet \exists\, a \bullet P(a,b)}{\vdash (\exists\, a \bullet \exists\, b \bullet P(a,b)) \Rightarrow (\exists\, b \bullet \exists\, a \bullet P(a,b))}\;\texttt{impI}}\;\texttt{exE}$$

Example: Existential Quantifiers Commute

$$(\exists\, a \bullet \exists\, b \bullet P(a, b)) \Rightarrow (\exists\, b \bullet \exists\, a \bullet P(a, b))$$

$$
\dfrac{x \notin fv(\exists\, b \bullet \exists\, a \bullet P(a,b)) \quad \dfrac{y \notin fv(\exists\, b \bullet \exists\, a \bullet P(a,b)) \quad \dfrac{\dfrac{\dfrac{}{P(x,y) \vdash P(x,y)}\,\texttt{asm}}{P(x,y) \vdash \exists\, a \bullet P(a,y)}\,\texttt{exI}}{P(x,y) \vdash \exists\, b \bullet \exists\, a \bullet P(a,b)}\,\texttt{exI}}{\exists\, b \bullet P(x,b) \vdash \exists\, b \bullet \exists\, a \bullet P(a,b)}\,\texttt{exE}}{\dfrac{\exists\, a \bullet \exists\, b \bullet P(a,b) \vdash \exists\, b \bullet \exists\, a \bullet P(a,b)}{\vdash (\exists\, a \bullet \exists\, b \bullet P(a,b)) \Rightarrow (\exists\, b \bullet \exists\, a \bullet P(a,b))}\,\texttt{impI}}\,\texttt{exE}
$$

Example: Existential Quantifiers Commute

$$(\exists\, a \bullet \exists\, b \bullet P(a,b)) \Rightarrow (\exists\, b \bullet \exists\, a \bullet P(a,b))$$

$$
\cfrac{x \notin fv(\exists\, b \bullet \exists\, a \bullet P(a,b)) \quad \cfrac{y \notin fv(\exists\, b \bullet \exists\, a \bullet P(a,b)) \quad \cfrac{\cfrac{\cfrac{\cfrac{-}{P(x,y) \vdash P(x,y)}\ \texttt{asm}}{P(x,y) \vdash \exists\, a \bullet P(a,y)}\ \texttt{exI}}{P(x,y) \vdash \exists\, b \bullet \exists\, a \bullet P(a,b)}\ \texttt{exI}}{\exists\, b \bullet P(x,b) \vdash \exists\, b \bullet \exists\, a \bullet P(a,b)}\ \texttt{exE}}{\exists\, a \bullet \exists\, b \bullet P(a,b) \vdash \exists\, b \bullet \exists\, a \bullet P(a,b)}\ \texttt{exE}}{\vdash (\exists\, a \bullet \exists\, b \bullet P(a,b)) \Rightarrow (\exists\, b \bullet \exists\, a \bullet P(a,b))}\ \texttt{impI}
$$

Example: Combining Quantifiers

- $(\forall a \bullet P(a) \Rightarrow \neg\, Q(a)) \Rightarrow (\neg\, \exists a \bullet P(a) \wedge Q(a))$
- Proof

Example: Combining Quantifiers

- $(\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a)) \Rightarrow (\neg\, \exists\, a \bullet P(a) \wedge Q(a))$
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$$\vdash (\forall a \bullet P(a) \Rightarrow \neg\, Q(a)) \Rightarrow (\neg\, \exists a \bullet P(a) \wedge Q(a))$$

Example: Combining Quantifiers

- $(\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))$
- Proof

$$\frac{}{\vdash (\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))}\ \texttt{impI}$$

Example: Combining Quantifiers

- $(\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))$
- Proof

$$\frac{\forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \neg \exists a \bullet P(a) \wedge Q(a)}{\vdash (\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))}\ \mathtt{impI}$$

Example: Combining Quantifiers

- $(\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a)) \Rightarrow (\neg\, \exists\, a \bullet P(a) \wedge Q(a))$
- Proof

$$\dfrac{\dfrac{}{\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \neg\, \exists\, a \bullet P(a) \wedge Q(a)}}{\vdash (\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a)) \Rightarrow (\neg\, \exists\, a \bullet P(a) \wedge Q(a))}\ \mathtt{impI}$$

Example: Combining Quantifiers

- $(\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \land Q(a))$
- Proof

$$\dfrac{\dfrac{}{\forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \neg \exists a \bullet P(a) \land Q(a)}\,\texttt{notI}}{\vdash (\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \land Q(a))}\,\texttt{impI}$$

Example: Combining Quantifiers

- $(\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \land Q(a))$
- Proof

$$\dfrac{\dfrac{\exists a \bullet P(a) \land Q(a), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \mathit{false}}{\forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \neg \exists a \bullet P(a) \land Q(a)}\ \mathtt{notI}}{\vdash (\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \land Q(a))}\ \mathtt{impI}$$

Example: Combining Quantifiers

- $(\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))$
- Proof

$$\dfrac{\dfrac{\dfrac{}{\exists a \bullet P(a) \wedge Q(a), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \mathit{false}}}{\forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \neg \exists a \bullet P(a) \wedge Q(a)}\ \texttt{notI}}{\vdash (\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))}\ \texttt{impI}$$

Example: Combining Quantifiers

- $(\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))$
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$$\dfrac{\dfrac{\dfrac{}{\exists a \bullet P(a) \wedge Q(a), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \mathit{false}}\ \texttt{exE}}{\forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \neg \exists a \bullet P(a) \wedge Q(a)}\ \texttt{notI}}{\vdash (\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))}\ \texttt{impI}$$

Example: Combining Quantifiers

- $(\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a)) \Rightarrow (\neg\, \exists\, a \bullet P(a) \wedge Q(a))$
- Proof

$$
\dfrac{\dfrac{x \notin \mathit{fv}(\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a), \mathit{false}) \quad \overline{\exists\, a \bullet P(a) \wedge Q(a), \forall\, a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \mathit{false}}\ \texttt{exE}}{\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \neg\, \exists\, a \bullet P(a) \wedge Q(a)}\ \texttt{notI}}{\vdash (\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a)) \Rightarrow (\neg\, \exists\, a \bullet P(a) \wedge Q(a))}\ \texttt{impI}
$$

Example: Combining Quantifiers

- $(\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a)) \Rightarrow (\neg\, \exists\, a \bullet P(a) \wedge Q(a))$
- Proof

$$
\dfrac{\dfrac{\dfrac{P(x) \wedge Q(x), \forall\, a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \mathit{false}}{\exists\, a \bullet P(a) \wedge Q(a), \forall\, a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \mathit{false}}\ \mathtt{exE}\quad x \notin \mathit{fv}(\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a), \mathit{false})}{\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \neg\, \exists\, a \bullet P(a) \wedge Q(a)}\ \mathtt{notI}}{\vdash (\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a)) \Rightarrow (\neg\, \exists\, a \bullet P(a) \wedge Q(a))}\ \mathtt{impI}
$$

Example: Combining Quantifiers

- $(\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a)) \Rightarrow (\neg\, \exists\, a \bullet P(a) \wedge Q(a))$
- Proof

$$
\dfrac{\dfrac{x \notin fv(\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a), \mathit{false}) \quad \dfrac{\dfrac{}{P(x) \wedge Q(x), \forall\, a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \mathit{false}}}{\exists\, a \bullet P(a) \wedge Q(a), \forall\, a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \mathit{false}}\,\texttt{exE}}{\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \neg\, \exists\, a \bullet P(a) \wedge Q(a)}\,\texttt{notI}}{\vdash (\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a)) \Rightarrow (\neg\, \exists\, a \bullet P(a) \wedge Q(a))}\,\texttt{impI}
$$

Example: Combining Quantifiers

- $(\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a)) \Rightarrow (\neg\, \exists\, a \bullet P(a) \wedge Q(a))$
- Proof

$$
\dfrac{\dfrac{x \notin fv(\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a), false) \quad \dfrac{P(x) \wedge Q(x), \forall\, a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash false}{\exists\, a \bullet P(a) \wedge Q(a), \forall\, a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash false}\,\texttt{exE}}{\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \neg\, \exists\, a \bullet P(a) \wedge Q(a)}\,\texttt{notI}}{\vdash (\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a)) \Rightarrow (\neg\, \exists\, a \bullet P(a) \wedge Q(a))}\,\texttt{impI}
$$

(The top line $P(x) \wedge Q(x), \forall\, a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash false$ has an open bar above it labelled $\texttt{conjE}$.)

Example: Combining Quantifiers

- $(\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a)) \Rightarrow (\neg\, \exists\, a \bullet P(a) \wedge Q(a))$
- Proof

$$\dfrac{x \notin fv(\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a), \mathit{false}) \quad \dfrac{\dfrac{P(x), Q(x), \forall\, a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \mathit{false}}{P(x) \wedge Q(x), \forall\, a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \mathit{false}}\ \texttt{conjE}}{\exists\, a \bullet P(a) \wedge Q(a), \forall\, a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \mathit{false}}\ \texttt{exE}}{\dfrac{\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \neg\, \exists\, a \bullet P(a) \wedge Q(a)}{\vdash (\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a)) \Rightarrow (\neg\, \exists\, a \bullet P(a) \wedge Q(a))}\ \texttt{impI}}\ \texttt{notI}$$

Example: Combining Quantifiers

- $(\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a)) \Rightarrow (\neg\, \exists\, a \bullet P(a) \wedge Q(a))$
- Proof

$$
\dfrac{\dfrac{x \notin fv(\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a), false) \quad \dfrac{\dfrac{}{P(x), Q(x), \forall\, a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash false}}{P(x) \wedge Q(x), \forall\, a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash false}\ \mathtt{conjE}}{\dfrac{\exists\, a \bullet P(a) \wedge Q(a), \forall\, a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash false}{\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \neg\, \exists\, a \bullet P(a) \wedge Q(a)}\ \mathtt{notI}}\ \mathtt{exE}}{\vdash (\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a)) \Rightarrow (\neg\, \exists\, a \bullet P(a) \wedge Q(a))}\ \mathtt{impI}
$$

Example: Combining Quantifiers

- $(\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a)) \Rightarrow (\neg\, \exists\, a \bullet P(a) \wedge Q(a))$
- Proof

$$
\dfrac{
x \notin fv(\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a), \mathit{false}) \quad
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{P(x), Q(x), \forall\, a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \mathit{false}}\;\texttt{rotate} \times 2
}{P(x) \wedge Q(x), \forall\, a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \mathit{false}}\;\texttt{conjE}
}{\exists\, a \bullet P(a) \wedge Q(a), \forall\, a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \mathit{false}}\;\texttt{exE}
}{\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \neg\, \exists\, a \bullet P(a) \wedge Q(a)}\;\texttt{notI}
}{}
$$

$$
\dfrac{\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \neg\, \exists\, a \bullet P(a) \wedge Q(a)}{\vdash (\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a)) \Rightarrow (\neg\, \exists\, a \bullet P(a) \wedge Q(a))}\;\texttt{impI}
$$

Example: Combining Quantifiers

- $(\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a)) \Rightarrow (\neg\, \exists\, a \bullet P(a) \wedge Q(a))$
- Proof

$$
\dfrac{
  x \notin fv(\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a), false) \quad
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{
          \forall\, a \bullet P(a) \Rightarrow \neg\, Q(a), P(x), Q(x) \vdash false
        }{
          P(x), Q(x), \forall\, a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash false
        }\ \texttt{rotate} \times 2
      }{
        P(x) \wedge Q(x), \forall\, a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash false
      }\ \texttt{conjE}
    }{
      \exists\, a \bullet P(a) \wedge Q(a), \forall\, a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash false
    }\ \texttt{exE}
  }{
    \forall\, a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \neg\, \exists\, a \bullet P(a) \wedge Q(a)
  }\ \texttt{notI}
}{
  \vdash (\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a)) \Rightarrow (\neg\, \exists\, a \bullet P(a) \wedge Q(a))
}\ \texttt{impI}
$$

Example: Combining Quantifiers

- $(\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a)) \Rightarrow (\neg\, \exists\, a \bullet P(a) \wedge Q(a))$
- Proof

$$
\dfrac{\dfrac{x \notin fv(\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a), false) \quad \dfrac{\dfrac{\dfrac{}{\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a), P(x), Q(x) \vdash false}}{P(x), Q(x), \forall\, a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash false}\ \texttt{rotate} \times 2}{P(x) \wedge Q(x), \forall\, a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash false}\ \texttt{conjE}}{\dfrac{\exists\, a \bullet P(a) \wedge Q(a), \forall\, a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash false}{\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \neg\, \exists\, a \bullet P(a) \wedge Q(a)}\ \texttt{notI}}\ \texttt{exE}}{\vdash (\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a)) \Rightarrow (\neg\, \exists\, a \bullet P(a) \wedge Q(a))}\ \texttt{impI}
$$

Example: Combining Quantifiers

- $(\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \land Q(a))$
- Proof

$$
\dfrac{\dfrac{x \notin fv(\forall a \bullet P(a) \Rightarrow \neg Q(a), false) \qquad \dfrac{\dfrac{\dfrac{}{\forall a \bullet P(a) \Rightarrow \neg Q(a), P(x), Q(x) \vdash false}\ \texttt{allE}}{P(x), Q(x), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash false}\ \texttt{rotate} \times 2}{P(x) \land Q(x), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash false}\ \texttt{conjE}}{\dfrac{\exists a \bullet P(a) \land Q(a), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash false}{\forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \neg \exists a \bullet P(a) \land Q(a)}\ \texttt{notI}}\ \texttt{exE}}{\vdash (\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \land Q(a))}\ \texttt{impI}
$$

Example: Combining Quantifiers

- $(\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))$
- Proof

$$\dfrac{\dfrac{x \notin fv(\forall a \bullet P(a) \Rightarrow \neg Q(a), false) \quad \dfrac{\dfrac{\dfrac{\dfrac{P(x) \Rightarrow \neg Q(x), P(x), Q(x) \vdash false}{\forall a \bullet P(a) \Rightarrow \neg Q(a), P(x), Q(x) \vdash false}\ \texttt{allE}}{P(x), Q(x), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash false}\ \texttt{rotate} \times 2}{P(x) \wedge Q(x), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash false}\ \texttt{conjE}}{\dfrac{\exists a \bullet P(a) \wedge Q(a), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash false}{\forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \neg \exists a \bullet P(a) \wedge Q(a)}\ \texttt{notI}}\ \texttt{exE}}{\vdash (\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))}\ \texttt{impI}$$

Example: Combining Quantifiers

- $(\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a)) \Rightarrow (\neg\, \exists\, a \bullet P(a) \land Q(a))$
- Proof

$$
\dfrac{\dfrac{x \notin fv(\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a), \mathit{false}) \quad \dfrac{\dfrac{\dfrac{\dfrac{\overline{P(x) \Rightarrow \neg\, Q(x), P(x), Q(x) \vdash \mathit{false}}}{\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a), P(x), Q(x) \vdash \mathit{false}}\ \texttt{allE}}{P(x), Q(x), \forall\, a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \mathit{false}}\ \texttt{rotate} \times 2}{P(x) \land Q(x), \forall\, a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \mathit{false}}\ \texttt{conjE}}{\exists\, a \bullet P(a) \land Q(a), \forall\, a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \mathit{false}}\ \texttt{exE}}{\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \neg\, \exists\, a \bullet P(a) \land Q(a)}\ \texttt{notI}}{\vdash (\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a)) \Rightarrow (\neg\, \exists\, a \bullet P(a) \land Q(a))}\ \texttt{impI}
$$

Example: Combining Quantifiers

- $(\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))$
- Proof

$$
\dfrac{\dfrac{x \notin \mathit{fv}(\forall a \bullet P(a) \Rightarrow \neg Q(a), \mathit{false}) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P(x) \Rightarrow \neg Q(x), P(x), Q(x) \vdash \mathit{false}}\ \texttt{impE}}{\forall a \bullet P(a) \Rightarrow \neg Q(a), P(x), Q(x) \vdash \mathit{false}}\ \texttt{allE}}{P(x), Q(x), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \mathit{false}}\ \texttt{rotate} \times 2}{P(x) \wedge Q(x), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \mathit{false}}\ \texttt{conjE}}{\exists a \bullet P(a) \wedge Q(a), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \mathit{false}}\ \texttt{exE}}{\dfrac{\forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \neg \exists a \bullet P(a) \wedge Q(a)}{\vdash (\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))}\ \texttt{impI}}\ \texttt{notI}
$$

Example: Combining Quantifiers

- $(\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a)) \Rightarrow (\neg\, \exists\, a \bullet P(a) \wedge Q(a))$
- Proof

$$
\dfrac{
  \dfrac{
    x \notin \mathit{fv}(\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a), \mathit{false}) \qquad
    \dfrac{
      \dfrac{
        \dfrac{
          \dfrac{
            \dfrac{P(x), Q(x) \vdash P(x) \qquad \neg\, Q(x), P(x), Q(x) \vdash \mathit{false}}
                  {P(x) \Rightarrow \neg\, Q(x), P(x), Q(x) \vdash \mathit{false}}\ \mathtt{impE}
          }{\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a), P(x), Q(x) \vdash \mathit{false}}\ \mathtt{allE}
        }{P(x), Q(x), \forall\, a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \mathit{false}}\ \mathtt{rotate} \times 2
      }{P(x) \wedge Q(x), \forall\, a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \mathit{false}}\ \mathtt{conjE}
    }{\exists\, a \bullet P(a) \wedge Q(a), \forall\, a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \mathit{false}}\ \mathtt{exE}
  }{\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \neg\, \exists\, a \bullet P(a) \wedge Q(a)}\ \mathtt{notI}
}{\vdash (\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a)) \Rightarrow (\neg\, \exists\, a \bullet P(a) \wedge Q(a))}\ \mathtt{impI}
$$

Example: Combining Quantifiers

- $(\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a)) \Rightarrow (\neg\, \exists\, a \bullet P(a) \wedge Q(a))$
- Proof

$$
\dfrac{
  x \notin fv(\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a), \mathit{false}) \quad
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{
          \dfrac{
            \dfrac{P(x), Q(x) \vdash P(x) \qquad \dfrac{}{\neg\, Q(x), P(x), Q(x) \vdash \mathit{false}}}{P(x) \Rightarrow \neg\, Q(x), P(x), Q(x) \vdash \mathit{false}}\;\mathtt{impE}
          }{\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a), P(x), Q(x) \vdash \mathit{false}}\;\mathtt{allE}
        }{P(x), Q(x), \forall\, a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \mathit{false}}\;\mathtt{rotate} \times 2
      }{P(x) \wedge Q(x), \forall\, a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \mathit{false}}\;\mathtt{conjE}
    }{\exists\, a \bullet P(a) \wedge Q(a), \forall\, a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \mathit{false}}\;\mathtt{exE}
  }{\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \neg\, \exists\, a \bullet P(a) \wedge Q(a)}\;\mathtt{notI}
}{\vdash (\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a)) \Rightarrow (\neg\, \exists\, a \bullet P(a) \wedge Q(a))}\;\mathtt{impI}
$$

Example: Combining Quantifiers

- $(\forall a \bullet P(a) \Rightarrow \neg\, Q(a)) \Rightarrow (\neg\, \exists a \bullet P(a) \wedge Q(a))$
- Proof

$$
\begin{prooftree}
\AxiomC{$P(x), Q(x) \vdash P(x)$}
\AxiomC{}
\RightLabel{$\mathtt{notE}$}
\UnaryInfC{$\neg\, Q(x), P(x), Q(x) \vdash \mathit{false}$}
\RightLabel{$\mathtt{impE}$}
\BinaryInfC{$P(x) \Rightarrow \neg\, Q(x), P(x), Q(x) \vdash \mathit{false}$}
\RightLabel{$\mathtt{allE}$}
\UnaryInfC{$\forall a \bullet P(a) \Rightarrow \neg\, Q(a), P(x), Q(x) \vdash \mathit{false}$}
\RightLabel{$\mathtt{rotate} \times 2$}
\UnaryInfC{$P(x), Q(x), \forall a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \mathit{false}$}
\RightLabel{$\mathtt{conjE}$}
\UnaryInfC{$P(x) \wedge Q(x), \forall a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \mathit{false}$}
\LeftLabel{$x \notin \mathit{fv}(\forall a \bullet P(a) \Rightarrow \neg\, Q(a), \mathit{false})$}
\RightLabel{$\mathtt{exE}$}
\UnaryInfC{$\exists a \bullet P(a) \wedge Q(a), \forall a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \mathit{false}$}
\RightLabel{$\mathtt{notI}$}
\UnaryInfC{$\forall a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \neg\, \exists a \bullet P(a) \wedge Q(a)$}
\RightLabel{$\mathtt{impI}$}
\UnaryInfC{$\vdash (\forall a \bullet P(a) \Rightarrow \neg\, Q(a)) \Rightarrow (\neg\, \exists a \bullet P(a) \wedge Q(a))$}
\end{prooftree}
$$

Example: Combining Quantifiers

- $(\forall a \bullet P(a) \Rightarrow \neg\, Q(a)) \Rightarrow (\neg\, \exists a \bullet P(a) \wedge Q(a))$
- Proof

$$
\dfrac{x \notin fv(\forall a \bullet P(a) \Rightarrow \neg\, Q(a), false) \qquad
\dfrac{\dfrac{\dfrac{\dfrac{P(x), Q(x) \vdash P(x) \qquad \dfrac{P(x), Q(x) \vdash Q(x)}{\neg\, Q(x), P(x), Q(x) \vdash false}\ \mathtt{notE}}{P(x) \Rightarrow \neg\, Q(x), P(x), Q(x) \vdash false}\ \mathtt{impE}}{\forall a \bullet P(a) \Rightarrow \neg\, Q(a), P(x), Q(x) \vdash false}\ \mathtt{allE}}{P(x), Q(x), \forall a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash false}\ \mathtt{rotate} \times 2}{P(x) \wedge Q(x), \forall a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash false}\ \mathtt{conjE}}{\dfrac{\dfrac{\exists a \bullet P(a) \wedge Q(a), \forall a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash false}{\forall a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \neg\, \exists a \bullet P(a) \wedge Q(a)}\ \mathtt{notI}}{\vdash (\forall a \bullet P(a) \Rightarrow \neg\, Q(a)) \Rightarrow (\neg\, \exists a \bullet P(a) \wedge Q(a))}\ \mathtt{impI}}\ \mathtt{exE}
$$

Example: Combining Quantifiers

- $(\forall a \bullet P(a) \Rightarrow \neg\, Q(a)) \Rightarrow (\neg\, \exists a \bullet P(a) \wedge Q(a))$
- Proof

$$
\dfrac{\dfrac{x \notin fv(\forall a \bullet P(a) \Rightarrow \neg\, Q(a), \mathit{false}) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(x), Q(x) \vdash P(x) \qquad \dfrac{\overline{P(x), Q(x) \vdash Q(x)}}{\neg\, Q(x), P(x), Q(x) \vdash \mathit{false}}\;\mathtt{notE}}{P(x) \Rightarrow \neg\, Q(x), P(x), Q(x) \vdash \mathit{false}}\;\mathtt{impE}}{\forall a \bullet P(a) \Rightarrow \neg\, Q(a), P(x), Q(x) \vdash \mathit{false}}\;\mathtt{allE}}{P(x), Q(x), \forall a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \mathit{false}}\;\mathtt{rotate} \times 2}{P(x) \wedge Q(x), \forall a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \mathit{false}}\;\mathtt{conjE}}{\exists a \bullet P(a) \wedge Q(a), \forall a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \mathit{false}}\;\mathtt{exE}}{\dfrac{\forall a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \neg\, \exists a \bullet P(a) \wedge Q(a)}{\vdash (\forall a \bullet P(a) \Rightarrow \neg\, Q(a)) \Rightarrow (\neg\, \exists a \bullet P(a) \wedge Q(a))}\;\mathtt{impI}}\;\mathtt{notI}
$$

Example: Combining Quantifiers

- $(\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a)) \Rightarrow (\neg\, \exists\, a \bullet P(a) \wedge Q(a))$
- Proof

$$
\dfrac{
  \dfrac{
    x \notin fv(\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a), false)
    \qquad
    \dfrac{
      \dfrac{
        \dfrac{
          \dfrac{
            \dfrac{
              P(x), Q(x) \vdash P(x)
              \qquad
              \dfrac{
                \dfrac{}{P(x), Q(x) \vdash Q(x)}\ \texttt{rotate}
              }{\neg\, Q(x), P(x), Q(x) \vdash false}\ \texttt{notE}
            }{P(x) \Rightarrow \neg\, Q(x), P(x), Q(x) \vdash false}\ \texttt{impE}
          }{\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a), P(x), Q(x) \vdash false}\ \texttt{allE}
        }{P(x), Q(x), \forall\, a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash false}\ \texttt{rotate} \times 2
      }{P(x) \wedge Q(x), \forall\, a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash false}\ \texttt{conjE}
    }{}
  }{
    \dfrac{\exists\, a \bullet P(a) \wedge Q(a), \forall\, a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash false}{\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \neg\, \exists\, a \bullet P(a) \wedge Q(a)}\ \texttt{notI}
  }\ \texttt{exE}
}{\vdash (\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a)) \Rightarrow (\neg\, \exists\, a \bullet P(a) \wedge Q(a))}\ \texttt{impI}
$$

Example: Combining Quantifiers

- $(\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a)) \Rightarrow (\neg\, \exists\, a \bullet P(a) \wedge Q(a))$
- Proof

$$\dfrac{Q(x), P(x) \vdash Q(x)}{P(x), Q(x) \vdash Q(x)}\;\texttt{rotate}$$

$$\dfrac{P(x), Q(x) \vdash Q(x)}{\neg\, Q(x), P(x), Q(x) \vdash \mathit{false}}\;\texttt{notE}$$

$$\dfrac{P(x), Q(x) \vdash P(x) \qquad \neg\, Q(x), P(x), Q(x) \vdash \mathit{false}}{P(x) \Rightarrow \neg\, Q(x), P(x), Q(x) \vdash \mathit{false}}\;\texttt{impE}$$

$$\dfrac{P(x) \Rightarrow \neg\, Q(x), P(x), Q(x) \vdash \mathit{false}}{\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a), P(x), Q(x) \vdash \mathit{false}}\;\texttt{allE}$$

$$\dfrac{\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a), P(x), Q(x) \vdash \mathit{false}}{P(x), Q(x), \forall\, a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \mathit{false}}\;\texttt{rotate} \times 2$$

$$\dfrac{P(x), Q(x), \forall\, a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \mathit{false}}{P(x) \wedge Q(x), \forall\, a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \mathit{false}}\;\texttt{conjE}$$

$$x \notin \mathit{fv}(\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a), \mathit{false}) \quad \dfrac{P(x) \wedge Q(x), \forall\, a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \mathit{false}}{\exists\, a \bullet P(a) \wedge Q(a), \forall\, a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \mathit{false}}\;\texttt{exE}$$

$$\dfrac{\exists\, a \bullet P(a) \wedge Q(a), \forall\, a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \mathit{false}}{\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \neg\, \exists\, a \bullet P(a) \wedge Q(a)}\;\texttt{notI}$$

$$\dfrac{\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \neg\, \exists\, a \bullet P(a) \wedge Q(a)}{\vdash (\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a)) \Rightarrow (\neg\, \exists\, a \bullet P(a) \wedge Q(a))}\;\texttt{impI}$$

Example: Combining Quantifiers

- $(\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))$
- Proof

$$
\begin{prooftree}
\AxiomC{$P(x), Q(x) \vdash P(x)$}
\AxiomC{}
\UnaryInfC{$Q(x), P(x) \vdash Q(x)$}
\RightLabel{\texttt{rotate}}
\UnaryInfC{$P(x), Q(x) \vdash Q(x)$}
\RightLabel{\texttt{notE}}
\UnaryInfC{$\neg Q(x), P(x), Q(x) \vdash \mathit{false}$}
\RightLabel{\texttt{impE}}
\BinaryInfC{$P(x) \Rightarrow \neg Q(x), P(x), Q(x) \vdash \mathit{false}$}
\RightLabel{\texttt{allE}}
\UnaryInfC{$\forall a \bullet P(a) \Rightarrow \neg Q(a), P(x), Q(x) \vdash \mathit{false}$}
\RightLabel{\texttt{rotate} $\times$ 2}
\UnaryInfC{$P(x), Q(x), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \mathit{false}$}
\RightLabel{\texttt{conjE}}
\UnaryInfC{$P(x) \wedge Q(x), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \mathit{false}$}
\LeftLabel{$x \notin \mathit{fv}(\forall a \bullet P(a) \Rightarrow \neg Q(a), \mathit{false})$}
\RightLabel{\texttt{exE}}
\UnaryInfC{$\exists a \bullet P(a) \wedge Q(a), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \mathit{false}$}
\RightLabel{\texttt{notI}}
\UnaryInfC{$\forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \neg \exists a \bullet P(a) \wedge Q(a)$}
\RightLabel{\texttt{impI}}
\UnaryInfC{$\vdash (\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))$}
\end{prooftree}
$$

Example: Combining Quantifiers

- $(\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a)) \Rightarrow (\neg\, \exists\, a \bullet P(a) \wedge Q(a))$
- Proof

$$
\dfrac{
  x \notin fv(\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a), \mathit{false})
  \qquad
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{
          P(x), Q(x) \vdash P(x)
          \qquad
          \dfrac{
            \dfrac{
              \dfrac{}{Q(x), P(x) \vdash Q(x)}\ \mathtt{asm}
            }{P(x), Q(x) \vdash Q(x)}\ \mathtt{rotate}
          }{\neg\, Q(x), P(x), Q(x) \vdash \mathit{false}}\ \mathtt{notE}
        }{P(x) \Rightarrow \neg\, Q(x), P(x), Q(x) \vdash \mathit{false}}\ \mathtt{impE}
      }{\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a), P(x), Q(x) \vdash \mathit{false}}\ \mathtt{allE}
    }{P(x), Q(x), \forall\, a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \mathit{false}}\ \mathtt{rotate} \times 2
  }{P(x) \wedge Q(x), \forall\, a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \mathit{false}}\ \mathtt{conjE}
}{
  \dfrac{
    \dfrac{\exists\, a \bullet P(a) \wedge Q(a), \forall\, a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \mathit{false}}{\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \neg\, \exists\, a \bullet P(a) \wedge Q(a)}\ \mathtt{notI}
  }{\vdash (\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a)) \Rightarrow (\neg\, \exists\, a \bullet P(a) \wedge Q(a))}\ \mathtt{impI}
}\ \mathtt{exE}
$$

Example: Combining Quantifiers

- $(\forall a \bullet P(a) \Rightarrow \neg\, Q(a)) \Rightarrow (\neg\, \exists a \bullet P(a) \wedge Q(a))$
- Proof

$$
\begin{prooftree}
\AxiomC{$P(x), Q(x) \vdash P(x)$}
\AxiomC{$-$}
\RightLabel{asm}
\UnaryInfC{$Q(x), P(x) \vdash Q(x)$}
\RightLabel{rotate}
\UnaryInfC{$P(x), Q(x) \vdash Q(x)$}
\RightLabel{notE}
\UnaryInfC{$\neg\, Q(x), P(x), Q(x) \vdash \mathit{false}$}
\RightLabel{impE}
\BinaryInfC{$P(x) \Rightarrow \neg\, Q(x), P(x), Q(x) \vdash \mathit{false}$}
\RightLabel{allE}
\UnaryInfC{$\forall a \bullet P(a) \Rightarrow \neg\, Q(a), P(x), Q(x) \vdash \mathit{false}$}
\RightLabel{rotate $\times$ 2}
\UnaryInfC{$P(x), Q(x), \forall a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \mathit{false}$}
\RightLabel{conjE}
\UnaryInfC{$P(x) \wedge Q(x), \forall a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \mathit{false}$}
\LeftLabel{$x \notin \mathit{fv}(\forall a \bullet P(a) \Rightarrow \neg\, Q(a), \mathit{false})$}
\RightLabel{exE}
\UnaryInfC{$\exists a \bullet P(a) \wedge Q(a), \forall a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \mathit{false}$}
\RightLabel{notI}
\UnaryInfC{$\forall a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \neg\, \exists a \bullet P(a) \wedge Q(a)$}
\RightLabel{impI}
\UnaryInfC{$\vdash (\forall a \bullet P(a) \Rightarrow \neg\, Q(a)) \Rightarrow (\neg\, \exists a \bullet P(a) \wedge Q(a))$}
\end{prooftree}
$$

Example: Combining Quantifiers

- $(\forall a \bullet P(a) \Rightarrow \neg\, Q(a)) \Rightarrow (\neg\, \exists a \bullet P(a) \wedge Q(a))$
- Proof

$$
\dfrac{
  x \notin fv(\forall a \bullet P(a) \Rightarrow \neg\, Q(a), \mathit{false})
  \qquad
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{
          \dfrac{
            \dfrac{}{P(x), Q(x) \vdash P(x)}
            \qquad
            \dfrac{
              \dfrac{
                \dfrac{-}{Q(x), P(x) \vdash Q(x)}\,\texttt{asm}
              }{P(x), Q(x) \vdash Q(x)}\,\texttt{rotate}
            }{\neg\, Q(x), P(x), Q(x) \vdash \mathit{false}}\,\texttt{notE}
          }{P(x) \Rightarrow \neg\, Q(x), P(x), Q(x) \vdash \mathit{false}}\,\texttt{impE}
        }{\forall a \bullet P(a) \Rightarrow \neg\, Q(a), P(x), Q(x) \vdash \mathit{false}}\,\texttt{allE}
      }{P(x), Q(x), \forall a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \mathit{false}}\,\texttt{rotate} \times 2
    }{P(x) \wedge Q(x), \forall a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \mathit{false}}\,\texttt{conjE}
  }{\exists a \bullet P(a) \wedge Q(a), \forall a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \mathit{false}}\,\texttt{exE}
}{
  \dfrac{\forall a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \neg\, \exists a \bullet P(a) \wedge Q(a)}{\vdash (\forall a \bullet P(a) \Rightarrow \neg\, Q(a)) \Rightarrow (\neg\, \exists a \bullet P(a) \wedge Q(a))}\,\texttt{impI}
}\,\texttt{notI}
$$

Example: Combining Quantifiers

- $(\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a)) \Rightarrow (\neg\, \exists\, a \bullet P(a) \wedge Q(a))$
- Proof

$$\dfrac{\dfrac{-}{Q(x), P(x) \vdash Q(x)}\ \texttt{asm}}{P(x), Q(x) \vdash Q(x)}\ \texttt{rotate}$$

$$\dfrac{P(x), Q(x) \vdash Q(x)}{\neg\, Q(x), P(x), Q(x) \vdash \mathit{false}}\ \texttt{notE}$$

$$\dfrac{\dfrac{}{P(x), Q(x) \vdash P(x)}\ \texttt{asm} \qquad \neg\, Q(x), P(x), Q(x) \vdash \mathit{false}}{P(x) \Rightarrow \neg\, Q(x), P(x), Q(x) \vdash \mathit{false}}\ \texttt{impE}$$

$$\dfrac{P(x) \Rightarrow \neg\, Q(x), P(x), Q(x) \vdash \mathit{false}}{\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a), P(x), Q(x) \vdash \mathit{false}}\ \texttt{allE}$$

$$\dfrac{\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a), P(x), Q(x) \vdash \mathit{false}}{P(x), Q(x), \forall\, a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \mathit{false}}\ \texttt{rotate} \times 2$$

$$\dfrac{P(x), Q(x), \forall\, a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \mathit{false}}{P(x) \wedge Q(x), \forall\, a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \mathit{false}}\ \texttt{conjE}$$

$$\dfrac{x \notin \mathit{fv}(\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a), \mathit{false}) \qquad P(x) \wedge Q(x), \forall\, a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \mathit{false}}{\exists\, a \bullet P(a) \wedge Q(a), \forall\, a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \mathit{false}}\ \texttt{exE}$$

$$\dfrac{\exists\, a \bullet P(a) \wedge Q(a), \forall\, a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \mathit{false}}{\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \neg\, \exists\, a \bullet P(a) \wedge Q(a)}\ \texttt{notI}$$

$$\dfrac{\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a) \vdash \neg\, \exists\, a \bullet P(a) \wedge Q(a)}{\vdash (\forall\, a \bullet P(a) \Rightarrow \neg\, Q(a)) \Rightarrow (\neg\, \exists\, a \bullet P(a) \wedge Q(a))}\ \texttt{impI}$$

Example: Combining Quantifiers

- $(\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \land Q(a))$
- Proof

$$
\dfrac{x \notin fv(\forall a \bullet P(a) \Rightarrow \neg Q(a), false) \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{-}{P(x), Q(x) \vdash P(x)}\,\texttt{asm} \quad \dfrac{\dfrac{\dfrac{-}{Q(x), P(x) \vdash Q(x)}\,\texttt{asm}}{P(x), Q(x) \vdash Q(x)}\,\texttt{rotate}}{\neg Q(x), P(x), Q(x) \vdash false}\,\texttt{notE}}{P(x) \Rightarrow \neg Q(x), P(x), Q(x) \vdash false}\,\texttt{impE}}{\forall a \bullet P(a) \Rightarrow \neg Q(a), P(x), Q(x) \vdash false}\,\texttt{allE}}{P(x), Q(x), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash false}\,\texttt{rotate} \times 2}{P(x) \land Q(x), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash false}\,\texttt{conjE}}{\exists a \bullet P(a) \land Q(a), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash false}\,\texttt{exE}}{\dfrac{\forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \neg \exists a \bullet P(a) \land Q(a)}{\vdash (\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \land Q(a))}\,\texttt{impI}}\,\texttt{notI}
$$

Summary

- Natural deduction rules for predicate calculus quantifiers.

Universal Introduction	safe
Universal Elimination	unsafe
Existential Introduction	unsafe
Existential Elimination	safe

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