

# Natural Deduction Predicate Calculus

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PROF

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# Overview

Natural Deduction Rules

Universal Quantification

Existential Quantification

# Outline

Natural Deduction Rules

Universal Quantification

Existential Quantification

$$x \notin fv(\Gamma) \frac{\Gamma \vdash P(x)}{\Gamma \vdash \forall a \bullet P(a)} \text{allI}$$

$$\frac{P(t), \Gamma \vdash R}{\forall a \bullet P(a), \Gamma \vdash R} \text{allE}$$

$$\frac{\Gamma \vdash P(t)}{\Gamma \vdash \exists a \bullet P(a)} \text{exI}$$

$$x \notin fv(\Gamma, Q) \frac{P(x), \Gamma \vdash Q}{\exists a \bullet P(a), \Gamma \vdash Q} \text{exE}$$

## Reasoning with Universal Quantification

- ▶ How can we reason with the universal quantifier?
- ▶ Take inspiration from the universal quantifier as generalised conjunction.
- ▶ Use and-introduction:  $\frac{\Gamma \vdash P \quad \Gamma \vdash Q}{\Gamma \vdash P \wedge Q}$  conjI.
- ▶ But this needs an infinite number of proofs to complete the proof tree!
- ▶ Insight: just prove a typical instance  $P(i)$  then generalise to all instances.
- ▶ Typical means arbitrary: make no assumption about which  $i \in \mathbb{N}$  we use.
- ▶ In general,  $i$  ranges over some set that is not necessarily the natural numbers.

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Existential Quantification

# Reasoning with Universal Quantification

- ▶ We formalise the notion of an **arbitrary choice**.
- ▶ We forbid the bound variable being mentioned in any hypothesis:  $x \notin fv(\Gamma)$ .
- ▶ So we are making no formal assumption about  $x$ .
- ▶ Universal introduction:

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- ▶ How do we deal with universal quantification as a hypothesis?

- ▶ Again, we take inspiration from the rule for conjunction.

- ▶ Use and-elimination:  $\frac{P, Q, \Gamma \vdash R}{P \wedge Q, \Gamma \vdash R} \text{ conjE}$

- ▶ But this adds an infinite number of hypotheses!

- ▶ Insight: Instead of adding all instances of  $P(i)$ , add a specific one:  $P(t)$ .

- ▶ Specialisation (contrast with generalisation).

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- ▶ Let's prove that  $(\forall a \bullet P(a)) \Rightarrow (\forall a \bullet P(a) \vee Q(a))$ .
- ▶ This proof uses both universal quantification rules.

$$\begin{array}{c} y \notin \text{fv}(P(x)) \quad \frac{P(x) \vdash P(y) \vee Q(y)}{P(x) \vdash \forall a \bullet P(a) \vee Q(a)} \text{allI} \\ \frac{\quad}{\forall a \bullet P(a) \vdash \forall a \bullet P(a) \vee Q(a)} \text{allE} \\ \hline \vdash (\forall a \bullet P(a)) \Rightarrow (\forall a \bullet P(a) \vee Q(a)) \text{impI} \end{array}$$

$$\begin{array}{c} \frac{\quad}{\forall a \bullet P(a) \vdash P(x)} \text{disjI1} \\ \frac{\quad}{\forall a \bullet P(a) \vdash P(x) \vee Q(x)} \text{allI} \\ \frac{\quad}{\forall a \bullet P(a) \vdash \forall a \bullet P(a) \vee Q(a)} \text{allI} \\ \hline \vdash (\forall a \bullet P(a)) \Rightarrow (\forall a \bullet P(a) \vee Q(a)) \text{impI} \end{array}$$

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$$\frac{\frac{\frac{x \notin \text{fv}(\forall a \bullet P(a)) \quad \frac{\overline{\forall a \bullet P(a) \vdash P(x)}}{\forall a \bullet P(a) \vdash P(x) \vee Q(x)} \text{disjI1}}{\forall a \bullet P(a) \vdash \forall a \bullet P(a) \vee Q(a)} \text{allI}}{\vdash (\forall a \bullet P(a)) \Rightarrow (\forall a \bullet P(a) \vee Q(a))} \text{impI}$$

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$$\begin{array}{c} \frac{\quad}{P(x) \vdash P(x)} \text{asm} \\ \frac{\quad}{\forall a \bullet P(a) \vdash P(x)} \text{allE} \\ \frac{\quad}{\forall a \bullet P(a) \vdash P(x) \vee Q(x)} \text{disjI1} \\ \text{x} \notin \text{fv}(\forall a \bullet P(a)) \quad \frac{\quad}{\forall a \bullet P(a) \vdash \forall a \bullet P(a) \vee Q(a)} \text{allI} \\ \frac{\quad}{\vdash (\forall a \bullet P(a)) \Rightarrow (\forall a \bullet P(a) \vee Q(a))} \text{impI} \end{array}$$



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# Example: Universal Quantification Distributes through Conjunction (L2R)

## Natural Deduction Rules

$$(\forall a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))$$

Proof

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Proof

$$\frac{\frac{}{\forall a \bullet P(a) \wedge Q(a) \vdash (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))} \text{conjI}}{\vdash (\forall a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))} \text{implI}$$

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Natural Deduction Rules

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$$\frac{\frac{\forall a \bullet P(a) \wedge Q(a) \vdash \forall a \bullet P(a) \quad \frac{x \notin \text{fv}(\Gamma_1) \quad \forall a \bullet P(a) \wedge Q(a) \vdash \forall a \bullet Q(a)}{\forall a \bullet P(a) \wedge Q(a) \vdash \forall a \bullet Q(a)} \text{allI}}{\forall a \bullet P(a) \wedge Q(a) \vdash (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))} \text{conjI}$$
$$\frac{\forall a \bullet P(a) \wedge Q(a) \vdash (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))}{\vdash (\forall a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))} \text{implI}$$

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$x \notin fv(\Gamma_1) \quad \frac{\forall a \bullet P(a) \wedge Q(a) \vdash Q(x)}{\forall a \bullet P(a) \wedge Q(a) \vdash \forall a \bullet Q(a)} \text{allI}$

# Example: Universal Quantification Distributes through Conjunction (L2R)

## Natural Deduction Rules

$$(\forall a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))$$

Proof

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# Example: Universal Quantification Distributes through Conjunction (L2R)

## Natural Deduction Rules

$$(\forall a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))$$

Proof

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$x \notin \text{fv}(\Gamma_1)$

# Example: Universal Quantification Distributes through Conjunction (L2R)

## Natural Deduction Rules

$$(\forall a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))$$

Proof

$$\frac{\frac{\frac{\forall a \bullet P(a) \wedge Q(a) \vdash \forall a \bullet P(a)}{\forall a \bullet P(a) \wedge Q(a) \vdash (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))} \quad \frac{\frac{\frac{P(x) \wedge Q(x) \vdash Q(x)}{\forall a \bullet P(a) \wedge Q(a) \vdash Q(x)} \text{allE} \quad \frac{\forall a \bullet P(a) \wedge Q(a) \vdash \forall a \bullet Q(a)}{\forall a \bullet P(a) \wedge Q(a) \vdash (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))} \text{allI}}{\forall a \bullet P(a) \wedge Q(a) \vdash (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))} \text{conjI}}{\vdash (\forall a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))} \text{implI}$$

## Example: Universal Quantification Distributes through Conjunction (L2R)

$$(\forall a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))$$

## Proof

$$\frac{\frac{\frac{\frac{P(x) \wedge Q(x) \vdash Q(x)}{\forall a \bullet P(a) \wedge Q(a) \vdash Q(a)} \text{allE}}{\forall a \bullet P(a) \wedge Q(a) \vdash \forall a \bullet Q(a)} \text{allI}}{\forall a \bullet P(a) \wedge Q(a) \vdash (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))} \text{conjI}}{\vdash (\forall a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))} \text{implI}$$

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$$(\forall a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))$$

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$$\frac{\begin{array}{c} \frac{}{\vdash P(x) \wedge Q(x)} \text{conjE} \\ \frac{}{\forall a \bullet P(a) \wedge Q(a) \vdash Q(x)} \text{allE} \\ \frac{\forall a \bullet P(a) \wedge Q(a) \vdash \forall a \bullet P(a) \quad x \notin fv(\Gamma_1)}{\forall a \bullet P(a) \wedge Q(a) \vdash \forall a \bullet Q(a)} \text{allI} \\ \frac{}{\vdash \forall a \bullet P(a) \wedge Q(a) \vdash \forall a \bullet Q(a)} \text{conjI} \end{array}}{\vdash (\forall a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))} \text{implI}$$

# Example: Universal Quantification Distributes through Conjunction (L2R)

## Natural Deduction Rules

$$(\forall a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))$$

Proof

$$\frac{\frac{\frac{\frac{P(x), Q(x) \vdash Q(x)}{P(x) \wedge Q(x) \vdash Q(x)} \text{conjE}}{\forall a \bullet P(a) \wedge Q(a) \vdash Q(x)} \text{allE}}{\forall a \bullet P(a) \wedge Q(a) \vdash \forall a \bullet Q(a)} \text{allI} \quad \frac{\forall a \bullet P(a) \wedge Q(a) \vdash \forall a \bullet P(a)}{\forall a \bullet P(a) \wedge Q(a) \vdash (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))} \text{conjI} \quad \frac{\frac{\frac{\frac{P(x), Q(x) \vdash Q(x)}{P(x) \wedge Q(x) \vdash Q(x)} \text{conjE}}{\forall a \bullet P(a) \wedge Q(a) \vdash Q(x)} \text{allE}}{\forall a \bullet P(a) \wedge Q(a) \vdash \forall a \bullet Q(a)} \text{allI} \quad \frac{\forall a \bullet P(a) \wedge Q(a) \vdash \forall a \bullet P(a)}{\forall a \bullet P(a) \wedge Q(a) \vdash (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))} \text{conjI} \quad \frac{\forall a \bullet P(a) \wedge Q(a) \vdash (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))}{\vdash (\forall a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))} \text{implI}$$

## Example: Universal Quantification Distributes through Conjunction (L2R)

## Natural Deduction Rules

$$(\forall a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))$$

## Proof

$$\begin{array}{c}
\frac{\frac{\frac{P(x), Q(x) \vdash Q(x)}{P(x) \wedge Q(x) \vdash Q(x)} \text{conjE}}{\forall a \bullet P(a) \wedge Q(a) \vdash Q(a)} \text{allE} \\
\frac{\forall a \bullet P(a) \wedge Q(a) \vdash \forall a \bullet P(a) \quad x \notin \text{fv}(\Gamma_1) \quad \frac{\forall a \bullet P(a) \wedge Q(a) \vdash \forall a \bullet Q(a)}{\forall a \bullet P(a) \wedge Q(a) \vdash (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))} \text{allI} \quad \text{conjI}}{\vdash (\forall a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))} \text{implI}
\end{array}$$

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## Natural Deduction Rules

$$(\forall a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))$$

## Proof

$$\frac{\frac{\frac{\frac{P(x), Q(x) \vdash Q(x)}{\text{thin}}}{P(x) \wedge Q(x) \vdash Q(x)} \text{conjE}}{\forall a \bullet P(a) \wedge Q(a) \vdash Q(a)} \text{allE}}{\frac{\forall a \bullet P(a) \wedge Q(a) \vdash \forall a \bullet P(a) \quad x \notin \text{fv}(\Gamma_1) \quad \forall a \bullet P(a) \wedge Q(a) \vdash \forall a \bullet Q(a)}{\text{allI}} \text{conjI}}{\frac{\forall a \bullet P(a) \wedge Q(a) \vdash (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))}{\vdash (\forall a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))} \text{implI}}$$

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$$(\forall a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))$$

Proof

$$\frac{\frac{\frac{\frac{Q(x) \vdash Q(x)}{P(x), Q(x) \vdash Q(x)} \text{thin}}{P(x) \wedge Q(x) \vdash Q(x)} \text{conjE}}{\forall a \bullet P(a) \wedge Q(a) \vdash Q(x)} \text{allE}}{\frac{\frac{\forall a \bullet P(a) \wedge Q(a) \vdash \forall a \bullet P(a)}{\forall a \bullet P(a) \wedge Q(a) \vdash \forall a \bullet P(a) \wedge \forall a \bullet Q(a)} \text{allI} \quad \frac{\frac{\frac{Q(x) \vdash Q(x)}{P(x), Q(x) \vdash Q(x)} \text{thin}}{P(x) \wedge Q(x) \vdash Q(x)} \text{conjE}}{\forall a \bullet P(a) \wedge Q(a) \vdash \forall a \bullet Q(a)} \text{allE}}{\frac{\forall a \bullet P(a) \wedge Q(a) \vdash (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))}{\vdash (\forall a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))} \text{conjI} \quad \text{implI}} \text{allI}$$



# Example: Universal Quantification Distributes through Conjunction (L2R)

## Natural Deduction Rules

$$(\forall a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))$$

Proof

$$\frac{\frac{\frac{\frac{\frac{Q(x) \vdash Q(x)}{P(x), Q(x) \vdash Q(x)} \text{thin}}{P(x) \wedge Q(x) \vdash Q(x)} \text{conjE}}{\forall a \bullet P(a) \wedge Q(a) \vdash Q(x)} \text{allE}}{\forall a \bullet P(a) \wedge Q(a) \vdash \forall a \bullet Q(a)} \text{allI} \quad \frac{\forall a \bullet P(a) \wedge Q(a) \vdash \forall a \bullet P(a) \quad x \notin \text{fv}(\Gamma_1)}{\forall a \bullet P(a) \wedge Q(a) \vdash (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))} \text{conjI}}{\vdash (\forall a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))} \text{implI}$$

# Example: Universal Quantification Distributes through Conjunction (L2R)

## Natural Deduction Rules

$$(\forall a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))$$

Proof

$$\frac{\frac{\frac{\frac{\frac{\frac{}{Q(x) \vdash Q(x)}{\text{asm}}}{P(x), Q(x) \vdash Q(x)}{\text{thin}}}{P(x) \wedge Q(x) \vdash Q(x)}{\text{conjE}}}{\forall a \bullet P(a) \wedge Q(a) \vdash Q(x)}{\text{allE}}}{\forall a \bullet P(a) \wedge Q(a) \vdash \forall a \bullet Q(a)}{\text{allI}}}{\forall a \bullet P(a) \wedge Q(a) \vdash \forall a \bullet P(a)}{\text{conjI}}}{\frac{\forall a \bullet P(a) \wedge Q(a) \vdash (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))}{\vdash (\forall a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))}}{\text{implI}} \quad x \notin \text{fv}(\Gamma_1)$$

# Example: Universal Quantification Distributes through Conjunction (L2R)

## Natural Deduction Rules

$$(\forall a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))$$

Proof

$$\frac{\frac{\frac{\frac{\frac{}{Q(x) \vdash Q(x)}{\text{asm}}}{P(x), Q(x) \vdash Q(x)}{\text{thin}}}{P(x) \wedge Q(x) \vdash Q(x)}{\text{conjE}}}{\forall a \bullet P(a) \wedge Q(a) \vdash Q(x)}{\text{allE}}}{\frac{\forall a \bullet P(a) \wedge Q(a) \vdash \forall a \bullet P(a)}{x \notin \text{fv}(\Gamma_1)} \quad \frac{\forall a \bullet P(a) \wedge Q(a) \vdash \forall a \bullet Q(a)}{\text{allI}}}{\frac{\forall a \bullet P(a) \wedge Q(a) \vdash \forall a \bullet P(a) \wedge \forall a \bullet Q(a)}{\text{conjI}}}{\frac{\forall a \bullet P(a) \wedge Q(a) \vdash (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))}{\text{implI}}}$$

# Example: Universal Quantification Distributes through Conjunction (L2R)

## Natural Deduction Rules

$$(\forall a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))$$

Proof

$$\frac{\frac{\frac{\frac{\frac{}{Q(x) \vdash Q(x)}{\text{asm}}}{P(x), Q(x) \vdash Q(x)}{\text{thin}}}{P(x) \wedge Q(x) \vdash Q(x)}{\text{conjE}}}{\forall a \bullet P(a) \wedge Q(a) \vdash Q(x)}{\text{allE}}}{\frac{\frac{\forall a \bullet P(a) \wedge Q(a) \vdash \forall a \bullet P(a)}{\text{conjI}} \quad \frac{x \notin \text{fv}(\Gamma_1) \quad \forall a \bullet P(a) \wedge Q(a) \vdash \forall a \bullet Q(a)}{\text{allI}}}{\forall a \bullet P(a) \wedge Q(a) \vdash (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))} \text{conjI}}{\vdash (\forall a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))} \text{implI}$$

# Example: Universal Quantification Distributes through Conjunction (L2R)

## Natural Deduction Rules

$$(\forall a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))$$

Proof

$$\frac{\frac{\frac{\frac{\frac{}{Q(x) \vdash Q(x)}{\text{asm}}}{P(x), Q(x) \vdash Q(x)}{\text{thin}}}{P(x) \wedge Q(x) \vdash Q(x)}{\text{conjE}}}{\forall a \bullet P(a) \wedge Q(a) \vdash Q(x)}{\text{allE}}}{\frac{\frac{\forall a \bullet P(a) \wedge Q(a) \vdash \forall a \bullet P(a)}{\text{allI}} \quad x \notin \text{fv}(\Gamma_1) \quad \frac{\forall a \bullet P(a) \wedge Q(a) \vdash \forall a \bullet Q(a)}{\text{allI}}}{\frac{\forall a \bullet P(a) \wedge Q(a) \vdash \forall a \bullet P(a) \wedge \forall a \bullet Q(a)}{\text{conjI}}}}{\frac{\forall a \bullet P(a) \wedge Q(a) \vdash (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))}{\text{implI}}}{\vdash (\forall a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))}$$

# Example: Universal Quantification Distributes through Conjunction (L2R)

## Natural Deduction Rules

$$(\forall a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))$$

Proof

$$\frac{\frac{\frac{\frac{\frac{\frac{}{Q(x) \vdash Q(x)}{\text{asm}}}{P(x), Q(x) \vdash Q(x)}{\text{thin}}}{P(x) \wedge Q(x) \vdash Q(x)}{\text{conjE}}}{\forall a \bullet P(a) \wedge Q(a) \vdash Q(x)}{\text{allE}}}{\forall a \bullet P(a) \wedge Q(a) \vdash \forall a \bullet Q(a)}{\text{allI}}}{\forall a \bullet P(a) \wedge Q(a) \vdash \forall a \bullet P(a)}{\text{conjI}}}{\vdash (\forall a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))}{\text{implI}}$$

$x \notin \text{fv}(\Gamma_2)$   $\forall a \bullet P(a) \wedge Q(a) \vdash \forall a \bullet P(a)$   $\text{allI}$   $x \notin \text{fv}(\Gamma_1)$

## Example: Universal Quantification Distributes through Conjunction (L2R)

## Natural Deduction Rules

$$(\forall a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))$$

### Proof

$$\frac{\frac{\frac{\frac{\frac{\frac{}{Q(x) \vdash Q(x)}{\text{asm}}}{P(x), Q(x) \vdash Q(x)}{\text{thin}}}{P(x) \wedge Q(x) \vdash Q(x)}{\text{conjE}}}{\forall a \bullet P(a) \wedge Q(a) \vdash Q(x)}{\text{allE}}}{\forall a \bullet P(a) \wedge Q(a) \vdash \forall a \bullet Q(a)}{\text{allI}}}{\frac{\frac{\frac{}{\forall a \bullet P(a) \wedge Q(a) \vdash P(x)}{\text{allI}}}{\forall a \bullet P(a) \wedge Q(a) \vdash \forall a \bullet P(a)}{\text{conjI}}}{\frac{\forall a \bullet P(a) \wedge Q(a) \vdash (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))}{\text{implI}}}{\vdash (\forall a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))}$$

## Example: Universal Quantification Distributes through Conjunction (L2R)

## Natural Deduction Rules

$$(\forall a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))$$

### Proof

$$\frac{\frac{\frac{\frac{\frac{\frac{}{Q(x) \vdash Q(x)}{\text{asm}}}{P(x), Q(x) \vdash Q(x)}{\text{thin}}}{P(x) \wedge Q(x) \vdash Q(x)}{\text{conjE}}}{\forall a \bullet P(a) \wedge Q(a) \vdash Q(x)}{\text{allE}}}{\forall a \bullet P(a) \wedge Q(a) \vdash \forall a \bullet Q(a)}{\text{allI}}}{\frac{\frac{\frac{}{\forall a \bullet P(a) \wedge Q(a) \vdash P(x)}{\text{allI}}}{\forall a \bullet P(a) \wedge Q(a) \vdash \forall a \bullet P(a)}{\text{conjI}}}{\frac{\forall a \bullet P(a) \wedge Q(a) \vdash (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))}{\text{implI}}}{\vdash (\forall a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))}$$



## Example: Universal Quantification Distributes through Conjunction (L2R)

## Natural Deduction Rules

$$(\forall a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))$$

### Proof

$$\begin{array}{c}
\frac{}{Q(x) \vdash Q(x)} \text{asm} \\
\frac{}{P(x), Q(x) \vdash Q(x)} \text{thin} \\
\frac{}{P(x) \wedge Q(x) \vdash Q(x)} \text{conjE} \\
\frac{}{\forall a \bullet P(a) \wedge Q(a) \vdash P(x)} \text{allE} \\
\frac{}{\forall a \bullet P(a) \wedge Q(a) \vdash \forall a \bullet P(a)} \text{allI} \quad x \notin \text{fv}(\Gamma_2) \\
\frac{}{\forall a \bullet P(a) \wedge Q(a) \vdash Q(x)} \text{allE} \\
\frac{}{\forall a \bullet P(a) \wedge Q(a) \vdash \forall a \bullet Q(a)} \text{allI} \quad x \notin \text{fv}(\Gamma_1) \\
\frac{}{\forall a \bullet P(a) \wedge Q(a) \vdash (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))} \text{conjI} \\
\frac{}{\vdash (\forall a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))} \text{implI}
\end{array}$$

## Example: Universal Quantification Distributes through Conjunction (L2R)

## Natural Deduction Rules

$$(\forall a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))$$

### Proof

[illegible]

## Example: Universal Quantification Distributes through Conjunction (L2R)

## Natural Deduction Rules

$$(\forall a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))$$

### Proof

$$\begin{array}{c}
\frac{\frac{\frac{\frac{\frac{\frac{}{Q(x) \vdash Q(x)}{\text{asm}}}{P(x), Q(x) \vdash Q(x)}{\text{thin}}}{P(x) \wedge Q(x) \vdash Q(x)}{\text{conjE}}}{\frac{\frac{\frac{}{P(x) \wedge Q(x) \vdash P(x)}}{\text{allE}}}{\forall a \bullet P(a) \wedge Q(a) \vdash P(a)}{\text{allI}}}{\frac{\frac{}{\forall a \bullet P(a) \wedge Q(a) \vdash \forall a \bullet P(a)}{\text{allI}}}{\forall a \bullet P(a) \wedge Q(a) \vdash (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))} \text{implI} \\
\vdash (\forall a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))
\end{array}$$

# Example: Universal Quantification Distributes through Conjunction (L2R)

## Natural Deduction Rules

$$(\forall a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))$$

Proof

$$\begin{array}{c}
 \frac{\frac{\frac{\frac{}{P(x) \wedge Q(x) \vdash P(x)}{\text{conjE}}}{\forall a \bullet P(a) \wedge Q(a) \vdash P(x)}{\text{allE}}}{\forall a \bullet P(a) \wedge Q(a) \vdash \forall a \bullet P(a)} \text{allI} \quad x \notin \text{fv}(\Gamma_2) \quad \frac{\frac{\frac{\frac{}{Q(x) \vdash Q(x)}{\text{asm}}}{P(x), Q(x) \vdash Q(x)}{\text{thin}}}{\frac{P(x) \wedge Q(x) \vdash Q(x)}{\text{conjE}}}}{\forall a \bullet P(a) \wedge Q(a) \vdash Q(x)} \text{allE}}{\forall a \bullet P(a) \wedge Q(a) \vdash \forall a \bullet Q(a)} \text{allI} \quad x \notin \text{fv}(\Gamma_1) \\
 \hline
 \frac{\forall a \bullet P(a) \wedge Q(a) \vdash (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))}{\vdash (\forall a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))} \text{conjI} \quad \text{implI}
 \end{array}$$

# Example: Universal Quantification Distributes through Conjunction (L2R)

## Natural Deduction Rules

$$(\forall a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))$$

Proof

$$\begin{array}{c}
 \frac{\frac{\frac{P(x), Q(x) \vdash P(x)}{P(x) \wedge Q(x) \vdash P(x)} \text{ conjE}}{\forall a \bullet P(a) \wedge Q(a) \vdash P(x)} \text{ allE}}{\forall a \bullet P(a) \wedge Q(a) \vdash \forall a \bullet P(a)} \text{ allI} \quad x \notin \text{fv}(\Gamma_2)
 \end{array}
 \quad
 \begin{array}{c}
 \frac{\frac{\frac{\frac{}{Q(x) \vdash Q(x)} \text{ asm}}{P(x), Q(x) \vdash Q(x)} \text{ thin}}{P(x) \wedge Q(x) \vdash Q(x)} \text{ conjE}}{\forall a \bullet P(a) \wedge Q(a) \vdash Q(x)} \text{ allE}}{\forall a \bullet P(a) \wedge Q(a) \vdash \forall a \bullet Q(a)} \text{ allI}
 \end{array}$$

$$\frac{\frac{\forall a \bullet P(a) \wedge Q(a) \vdash \forall a \bullet P(a) \quad \forall a \bullet P(a) \wedge Q(a) \vdash \forall a \bullet Q(a)}{\forall a \bullet P(a) \wedge Q(a) \vdash (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))} \text{ conjI}}{\vdash (\forall a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))} \text{ implI}$$

## Example: Universal Quantification Distributes through Conjunction (L2R)

## Natural Deduction Rules

$$(\forall a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))$$

### Proof

$$\begin{array}{c}
\frac{\frac{\frac{\frac{}{P(x), Q(x) \vdash P(x)}}{P(x) \wedge Q(x) \vdash P(x)} \text{conjE}}{\forall a \bullet P(a) \wedge Q(a) \vdash P(x)} \text{allE}}{\forall a \bullet P(a) \wedge Q(a) \vdash \forall a \bullet P(a)} \text{allI} \quad x \notin \text{fv}(\Gamma_2)
\end{array}
\quad
\begin{array}{c}
\frac{\frac{\frac{\frac{}{Q(x) \vdash Q(x)} \text{asm}}{P(x), Q(x) \vdash Q(x)} \text{thin}}{P(x) \wedge Q(x) \vdash Q(x)} \text{conjE}}{\forall a \bullet P(a) \wedge Q(a) \vdash Q(x)} \text{allE}}{\forall a \bullet P(a) \wedge Q(a) \vdash \forall a \bullet Q(a)} \text{allI} \quad x \notin \text{fv}(\Gamma_1)
\end{array}
\quad
\frac{\frac{\frac{\frac{}{\forall a \bullet P(a) \wedge Q(a) \vdash (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))}}{\vdash (\forall a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))} \text{implI}}{\vdash (\forall a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))} \text{conjI}$$

## Example: Universal Quantification Distributes through Conjunction (L2R)

## Natural Deduction Rules

$$(\forall a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))$$

### Proof

[illegible]

# Example: Universal Quantification Distributes through Conjunction (L2R)

## Natural Deduction Rules

$$(\forall a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))$$

Proof

$$\begin{array}{c}
 \frac{\frac{\frac{-}{P(x), Q(x) \vdash P(x)}{\text{asm}}}{\frac{P(x) \wedge Q(x) \vdash P(x)}{\text{conjE}}} \text{allE} \quad \frac{\frac{\frac{-}{Q(x) \vdash Q(x)}{\text{asm}}}{\frac{P(x), Q(x) \vdash Q(x)}{\text{thin}}} \text{conjE} \\
 \frac{\frac{\frac{\frac{-}{P(x) \wedge Q(x) \vdash P(x)}{\text{conjE}}} \text{allE}}{\forall a \bullet P(a) \wedge Q(a) \vdash P(x)} \text{allI} \quad \frac{\frac{\frac{\frac{-}{P(x) \wedge Q(x) \vdash Q(x)}{\text{conjE}}} \text{allE}}{\forall a \bullet P(a) \wedge Q(a) \vdash Q(x)} \text{allI} \\
 \frac{\frac{\frac{\frac{-}{\forall a \bullet P(a) \wedge Q(a) \vdash P(x)}{\text{allI}}}{\forall a \bullet P(a) \wedge Q(a) \vdash \forall a \bullet P(a)} \text{conjI} \quad \frac{\frac{\frac{\frac{-}{\forall a \bullet P(a) \wedge Q(a) \vdash Q(x)}{\text{allI}}}{\forall a \bullet P(a) \wedge Q(a) \vdash \forall a \bullet Q(a)} \text{conjI} \\
 \frac{\frac{\frac{\frac{-}{\forall a \bullet P(a) \wedge Q(a) \vdash (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))}{\text{conjI}}}{\vdash (\forall a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))} \text{implI}
 \end{array}$$



# Example: Universal Quantification Distributes through Conjunction (L2R)

## Natural Deduction Rules

$$(\forall a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))$$

Proof

$$\begin{array}{c}
 \frac{\frac{\frac{-}{P(x), Q(x) \vdash P(x)}{\text{asm}}}{\frac{P(x) \wedge Q(x) \vdash P(x)}{\text{conjE}}} \text{allE} \quad \frac{\frac{\frac{-}{Q(x) \vdash Q(x)}{\text{asm}}}{\frac{P(x), Q(x) \vdash Q(x)}{\text{thin}}} \text{conjE} \\
 \frac{\frac{\frac{\frac{-}{P(x) \wedge Q(x) \vdash P(x)}{\text{conjE}}} \text{allE}}{\forall a \bullet P(a) \wedge Q(a) \vdash P(x)} \text{allI} \quad \frac{\frac{\frac{\frac{-}{P(x) \wedge Q(x) \vdash Q(x)}{\text{conjE}}} \text{allE}}{\forall a \bullet P(a) \wedge Q(a) \vdash Q(x)} \text{allI} \\
 \frac{\frac{\frac{\frac{-}{\forall a \bullet P(a) \wedge Q(a) \vdash P(x)}{\text{allI}}}{\forall a \bullet P(a) \wedge Q(a) \vdash \forall a \bullet P(a)} \text{conjI} \quad \frac{\frac{\frac{\frac{-}{\forall a \bullet P(a) \wedge Q(a) \vdash Q(x)}{\text{allI}}}{\forall a \bullet P(a) \wedge Q(a) \vdash \forall a \bullet Q(a)} \text{conjI} \\
 \frac{\frac{\frac{\frac{-}{\forall a \bullet P(a) \wedge Q(a) \vdash (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))}{\text{conjI}}}{\vdash (\forall a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))} \text{implI}
 \end{array}$$

$x \notin \text{fv}(\Gamma_1)$

# Example: Universal Quantification Distributes through Conjunction (L2R)

## Natural Deduction Rules

$$(\forall a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))$$

Proof

$$\begin{array}{c}
 \frac{\frac{\frac{-}{P(x), Q(x) \vdash P(x)}{\text{asm}}}{\frac{P(x) \wedge Q(x) \vdash P(x)}{\text{conjE}}} \quad \frac{\frac{\frac{-}{Q(x) \vdash Q(x)}{\text{asm}}}{\frac{P(x), Q(x) \vdash Q(x)}{\text{thin}}} \quad \frac{\frac{P(x) \wedge Q(x) \vdash Q(x)}{\text{conjE}}}{\frac{P(x) \wedge Q(x) \vdash Q(x)}{\text{allE}}} \\
 \frac{\frac{\frac{\frac{-}{\forall a \bullet P(a) \wedge Q(a) \vdash P(x)}{\text{allE}}}{\frac{\forall a \bullet P(a) \wedge Q(a) \vdash P(x)}{\text{allI}}} \quad \frac{\frac{\frac{\frac{-}{\forall a \bullet P(a) \wedge Q(a) \vdash Q(x)}{\text{allE}}}{\frac{\forall a \bullet P(a) \wedge Q(a) \vdash Q(x)}{\text{allI}}} \quad \frac{\frac{\frac{-}{\forall a \bullet P(a) \wedge Q(a) \vdash (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))}{\text{conjI}}} \\
 \frac{\frac{\frac{\frac{-}{\forall a \bullet P(a) \wedge Q(a) \vdash (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))}{\text{conjI}}}{\frac{\vdash (\forall a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))}{\text{implI}}}
 \end{array}$$

$$x \notin \text{fv}(\Gamma_1) = x \notin \text{fv}(\Gamma_2)$$

# Example: Universal Quantification Distributes through Conjunction (L2R)

Natural Deduction Rules

$$(\forall a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))$$

Proof

$$\begin{array}{c}
 \frac{\frac{\frac{\frac{}{} \text{asm}}{P(x), Q(x) \vdash P(x)} \text{conjE}}{P(x) \wedge Q(x) \vdash P(x)} \text{allE}}{\forall a \bullet P(a) \wedge Q(a) \vdash P(x)} \text{allI} \quad x \notin \text{fv}(\Gamma_2) \quad \frac{\frac{\frac{\frac{}{} \text{asm}}{Q(x) \vdash Q(x)} \text{thin}}{P(x), Q(x) \vdash Q(x)} \text{conjE}}{P(x) \wedge Q(x) \vdash Q(x)} \text{allE}}{\forall a \bullet P(a) \wedge Q(a) \vdash Q(x)} \text{allI} \quad x \notin \text{fv}(\Gamma_1) \\
 \hline
 \frac{\forall a \bullet P(a) \wedge Q(a) \vdash \forall a \bullet P(a) \quad \forall a \bullet P(a) \wedge Q(a) \vdash \forall a \bullet Q(a)}{\forall a \bullet P(a) \wedge Q(a) \vdash (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))} \text{conjI} \\
 \hline
 \frac{}{\vdash (\forall a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))} \text{implI}
 \end{array}$$

$$x \notin \text{fv}(\Gamma_1) = x \notin \text{fv}(\Gamma_2) = x \notin \text{fv}(\forall a \bullet P(a) \wedge Q(a))$$

# Example: Universal Quantification Distributes through Conjunction (L2R)

Natural Deduction Rules

$$(\forall a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))$$

Proof

$$\begin{array}{c}
 \frac{\frac{\frac{\frac{}{P(x), Q(x) \vdash P(x)}{\text{asm}}}{P(x) \wedge Q(x) \vdash P(x)}{\text{conjE}}}{\forall a \bullet P(a) \wedge Q(a) \vdash P(x)}{\text{allE}}}{\forall a \bullet P(a) \wedge Q(a) \vdash \forall a \bullet P(a)}{\text{allI}} \quad x \notin \text{fv}(\Gamma_2)
 \end{array}
 \quad
 \begin{array}{c}
 \frac{\frac{\frac{\frac{}{Q(x) \vdash Q(x)}{\text{asm}}}{P(x), Q(x) \vdash Q(x)}{\text{thin}}}{P(x) \wedge Q(x) \vdash Q(x)}{\text{conjE}}}{\forall a \bullet P(a) \wedge Q(a) \vdash Q(x)}{\text{allE}}}{\forall a \bullet P(a) \wedge Q(a) \vdash \forall a \bullet Q(a)}{\text{allI}} \quad x \notin \text{fv}(\Gamma_1)
 \end{array}$$

$$\frac{\frac{\frac{\frac{}{\forall a \bullet P(a) \wedge Q(a) \vdash \forall a \bullet P(a)}{\text{allI}} \quad x \notin \text{fv}(\Gamma_2) \quad \frac{\frac{\frac{}{\forall a \bullet P(a) \wedge Q(a) \vdash \forall a \bullet Q(a)}{\text{allI}} \quad x \notin \text{fv}(\Gamma_1)}{\text{conjI}}}{\forall a \bullet P(a) \wedge Q(a) \vdash (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))}}{\vdash (\forall a \bullet P(a) \wedge Q(a)) \Rightarrow (\forall a \bullet P(a)) \wedge (\forall a \bullet Q(a))}}{\text{implI}}$$

$$x \notin \text{fv}(\Gamma_1) = x \notin \text{fv}(\Gamma_2) = x \notin \text{fv}(\forall a \bullet P(a) \wedge Q(a)) = \text{true}$$

# Outline

Natural Deduction Rules

Universal Quantification

Existential Quantification

# Existential Quantification

- ▶ How do we reason with the existential quantifier?
- ▶ It's a generalisation of disjunction, so take inspiration from the `disjI` rules

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- ▶ How do we reason with an **existential hypothesis**?  $\exists a \bullet P(a) \vdash Q$
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# Existential Elimination

- ▶ Existential elimination is the dual of universal introduction.

- ▶ Existential elimination: 
$$x \notin fv(\Gamma, Q) \frac{P(x), \Gamma \vdash Q}{\exists a \bullet P(a), \Gamma \vdash Q} \text{exE}$$

- ▶ Like universal introduction, there is an important side condition.
- ▶ The fixed variable must be arbitrary:  $x \notin fv(\Gamma, Q)$ .
- ▶ Existential elimination allows us to derive an arbitrary instance of a predicate.
- ▶ We can then derive conclusions from that arbitrary instance.
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- ▶ We prove something about the genuinely typical case.

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### ► Convention

$\exists a \bullet P(a)$     versus     $\exists a \bullet P$

$P(a)$  means  $a$  is free for substitution

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### ► Example    $(\exists a \bullet P) \Leftrightarrow P$

Proof

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*Note: In the original image, the condition  $x \notin \text{fv}(P)$  is written to the left of the first derivation. The rule 'exE' is also labeled in green in the original image.*

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$$\begin{array}{c} \frac{\frac{\frac{}{P \vdash P} \text{asm}}{\exists a \bullet P \vdash P} \text{exE}}{\vdash (\exists a \bullet P) \Rightarrow P} \text{impI} \quad \frac{\frac{\frac{}{P \vdash P} \text{asm}}{P \vdash \exists a \bullet P} \text{exI}}{\vdash P \Rightarrow (\exists a \bullet P)} \text{impI} \\ \hline \vdash (\exists a \bullet P) \Leftrightarrow P \quad \text{iffI} \end{array}$$

*Note: In the original image, the condition  $x \notin \text{fv}(P)$  is written to the left of the first derivation. The rule labels (asm, exE, exI, impI, iffI) are in green.*

## Example: Reasoning with Existential Quantification

$$(\exists a \bullet P(a) \wedge Q(a)) \Rightarrow (\exists a \bullet P(a))$$



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## Example: Existential Quantifiers Commute

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Natural Deduction Rules

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# Example: Existential Quantifiers Commute

$$(\exists a \bullet \exists b \bullet P(a, b)) \Rightarrow (\exists b \bullet \exists a \bullet P(a, b))$$

$$\begin{array}{c} \frac{\frac{\frac{\frac{}{P(x, y) \vdash P(x, y)}{\text{asm}}}{P(x, y) \vdash \exists a \bullet P(a, y)}{\text{exI}}}{P(x, y) \vdash \exists b \bullet \exists a \bullet P(a, b)}{\text{exI}}}{\frac{y \notin fv(\exists b \bullet \exists a \bullet P(a, b))}{\frac{\exists b \bullet P(x, b) \vdash \exists b \bullet \exists a \bullet P(a, b)}{\text{exE}}}}{\frac{x \notin fv(\exists b \bullet \exists a \bullet P(a, b))}{\frac{\exists a \bullet \exists b \bullet P(a, b) \vdash \exists b \bullet \exists a \bullet P(a, b)}{\text{exE}}}}{\vdash (\exists a \bullet \exists b \bullet P(a, b)) \Rightarrow (\exists b \bullet \exists a \bullet P(a, b))}{\text{impI}} \end{array}$$



# Example: Existential Quantifiers Commute

Natural Deduction Rules

$$(\exists a \bullet \exists b \bullet P(a, b)) \Rightarrow (\exists b \bullet \exists a \bullet P(a, b))$$

$$\begin{array}{c}
 \frac{}{P(x, y) \vdash P(x, y)} \text{asm} \\
 \frac{P(x, y) \vdash P(x, y)}{P(x, y) \vdash \exists a \bullet P(a, y)} \text{exI} \\
 \frac{P(x, y) \vdash \exists a \bullet P(a, y)}{P(x, y) \vdash \exists b \bullet \exists a \bullet P(a, b)} \text{exI} \\
 \frac{y \notin fv(\exists b \bullet \exists a \bullet P(a, b)) \quad P(x, y) \vdash \exists b \bullet \exists a \bullet P(a, b)}{\exists b \bullet P(x, b) \vdash \exists b \bullet \exists a \bullet P(a, b)} \text{exE} \\
 \frac{x \notin fv(\exists b \bullet \exists a \bullet P(a, b)) \quad \exists b \bullet P(x, b) \vdash \exists b \bullet \exists a \bullet P(a, b)}{\exists a \bullet \exists b \bullet P(a, b) \vdash \exists b \bullet \exists a \bullet P(a, b)} \text{exE} \\
 \frac{\exists a \bullet \exists b \bullet P(a, b) \vdash \exists b \bullet \exists a \bullet P(a, b)}{\vdash (\exists a \bullet \exists b \bullet P(a, b)) \Rightarrow (\exists b \bullet \exists a \bullet P(a, b))} \text{impI}
 \end{array}$$

## Example: Combining Quantifiers

- ▶  $(\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))$
- ▶ Proof

## Example: Combining Quantifiers

►  $(\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))$

► Proof

## Example: Combining Quantifiers

►  $(\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))$

► Proof

## Example: Combining Quantifiers

►  $(\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))$

► Proof

$$\vdash (\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))$$

## Example: Combining Quantifiers

►  $(\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))$

► Proof

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$$\vdash (\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))$$

## Example: Combining Quantifiers

►  $(\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))$

► Proof

$$\frac{}{\vdash (\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))} \text{impI}$$

## Example: Combining Quantifiers

►  $(\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))$

► Proof

$$\frac{\forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \neg \exists a \bullet P(a) \wedge Q(a)}{\vdash (\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))} \text{impI}$$



## Example: Combining Quantifiers

►  $(\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))$

► Proof

$$\frac{\forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \neg \exists a \bullet P(a) \wedge Q(a)}{\vdash (\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))} \text{impI}$$

## Example: Combining Quantifiers

►  $(\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))$

► Proof

$$\frac{\frac{}{\forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \neg \exists a \bullet P(a) \wedge Q(a)} \text{notI}}{\vdash (\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))} \text{impI}$$

## Example: Combining Quantifiers

►  $(\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))$

► Proof

$$\frac{\frac{\exists a \bullet P(a) \wedge Q(a), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}}{\forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \neg \exists a \bullet P(a) \wedge Q(a)} \text{notI}}{\vdash (\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))} \text{impI}$$

## Example: Combining Quantifiers

►  $(\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))$

► Proof

$$\frac{\frac{\frac{}{\exists a \bullet P(a) \wedge Q(a), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}}{\forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \neg \exists a \bullet P(a) \wedge Q(a)} \text{notI}}{\vdash (\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))} \text{impI}$$

## Example: Combining Quantifiers

►  $(\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))$

► Proof

$$\frac{\frac{\frac{}{\exists a \bullet P(a) \wedge Q(a), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}}{\forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \neg \exists a \bullet P(a) \wedge Q(a)} \text{exE, notI}}{\vdash (\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))} \text{impI}$$

# Example: Combining Quantifiers

►  $(\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))$

► Proof

$$\frac{\frac{\frac{x \notin \text{fv}(\forall a \bullet P(a) \Rightarrow \neg Q(a), \text{false})}{\exists a \bullet P(a) \wedge Q(a), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}} \text{exE}}{\forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \neg \exists a \bullet P(a) \wedge Q(a)} \text{notI}}{\vdash (\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))} \text{impI}$$

## Example: Combining Quantifiers

►  $(\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))$

► Proof

$$\begin{array}{c} x \notin \text{fv}(\forall a \bullet P(a) \Rightarrow \neg Q(a), \text{false}) \quad \frac{P(x) \wedge Q(x), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}}{\exists a \bullet P(a) \wedge Q(a), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}} \text{exE} \\ \frac{\exists a \bullet P(a) \wedge Q(a), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}}{\forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \neg \exists a \bullet P(a) \wedge Q(a)} \text{notI} \\ \frac{\forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \neg \exists a \bullet P(a) \wedge Q(a)}{\vdash (\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))} \text{impI} \end{array}$$

# Example: Combining Quantifiers

►  $(\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))$

► Proof

$$\begin{array}{c} x \notin \text{fv}(\forall a \bullet P(a) \Rightarrow \neg Q(a), \text{false}) \quad \frac{\frac{P(x) \wedge Q(x), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}}{\exists a \bullet P(a) \wedge Q(a), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}} \text{exE}}{\frac{\forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \neg \exists a \bullet P(a) \wedge Q(a)}{\vdash (\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))} \text{notI}} \text{impI} \end{array}$$



# Example: Combining Quantifiers

►  $(\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))$

► Proof

$$\begin{array}{c} \frac{\frac{\frac{\frac{P(x) \wedge Q(x), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}}{\exists a \bullet P(a) \wedge Q(a), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}} \text{conjE}}{\forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \neg \exists a \bullet P(a) \wedge Q(a)} \text{exE}}{\vdash (\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))} \text{notI} \\ \text{implI} \end{array}$$

$x \notin \text{fv}(\forall a \bullet P(a) \Rightarrow \neg Q(a), \text{false})$

## Example: Combining Quantifiers

►  $(\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))$

► Proof

$$\begin{array}{c} \frac{P(x), Q(x), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}}{P(x) \wedge Q(x), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}} \text{conjE} \\ \frac{\frac{P(x) \wedge Q(x), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}}{\exists a \bullet P(a) \wedge Q(a), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}} \text{exE}}{\forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \neg \exists a \bullet P(a) \wedge Q(a)} \text{notI} \\ \frac{\forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \neg \exists a \bullet P(a) \wedge Q(a)}{\vdash (\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))} \text{impI} \end{array}$$

$x \notin \text{fv}(\forall a \bullet P(a) \Rightarrow \neg Q(a), \text{false})$

## Example: Combining Quantifiers

►  $(\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))$

► Proof

$$\begin{array}{c} \frac{\frac{\frac{\frac{\frac{}{P(x), Q(x), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}}{P(x) \wedge Q(x), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}}{\exists a \bullet P(a) \wedge Q(a), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}}{\forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \neg \exists a \bullet P(a) \wedge Q(a)} \text{conjE} \quad \text{exE} \quad \text{notI}}{\vdash (\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))} \text{impI} \\ x \notin \text{fv}(\forall a \bullet P(a) \Rightarrow \neg Q(a), \text{false}) \end{array}$$

## Natural Deduction Rules

- [illegible]

## Example: Combining Quantifiers

►  $(\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))$

► Proof

$$\begin{array}{c} \frac{\frac{\frac{\frac{\frac{\forall a \bullet P(a) \Rightarrow \neg Q(a), P(x), Q(x) \vdash \text{false}}{P(x), Q(x), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}}{P(x) \wedge Q(x), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}}{\exists a \bullet P(a) \wedge Q(a), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}}}{\forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \neg \exists a \bullet P(a) \wedge Q(a)}}{x \notin \text{fv}(\forall a \bullet P(a) \Rightarrow \neg Q(a), \text{false}) \vdash (\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))} \begin{array}{l} \text{rotate } \times 2 \\ \text{conjE} \\ \text{exE} \\ \text{notI} \\ \text{impI} \end{array} \end{array}$$

## Natural Deduction Rules

- $$x \notin fv(\forall a \bullet P(a) \Rightarrow \neg Q(a), false)$$

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## Natural Deduction Rules

- $$\begin{array}{c}
\frac{}{\forall a \bullet P(a) \Rightarrow \neg Q(a), P(x), Q(x) \vdash \text{false}} \text{allE} \\
\frac{}{P(x), Q(x), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}} \text{rotate} \times 2 \\
\frac{}{P(x) \wedge Q(x), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}} \text{conjE} \\
x \notin \text{fv}(\forall a \bullet P(a) \Rightarrow \neg Q(a), \text{false}) \quad \frac{}{\exists a \bullet P(a) \wedge Q(a), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}} \text{exE} \\
\frac{}{\forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \neg \exists a \bullet P(a) \wedge Q(a)} \text{notI} \\
\hline
\vdash (\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a)) \quad \text{impI}
\end{array}$$

## Example: Combining Quantifiers

►  $(\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))$

► Proof

$$\begin{array}{c} \frac{P(x) \Rightarrow \neg Q(x), P(x), Q(x) \vdash \text{false}}{\forall a \bullet P(a) \Rightarrow \neg Q(a), P(x), Q(x) \vdash \text{false}} \text{allE} \\ \frac{\forall a \bullet P(a) \Rightarrow \neg Q(a), P(x), Q(x) \vdash \text{false}}{P(x), Q(x), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}} \text{rotate} \times 2 \\ \frac{P(x), Q(x), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}}{P(x) \wedge Q(x), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}} \text{conjE} \\ \frac{P(x) \wedge Q(x), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}}{\exists a \bullet P(a) \wedge Q(a), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}} \text{exE} \\ \frac{\exists a \bullet P(a) \wedge Q(a), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}}{\forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \neg \exists a \bullet P(a) \wedge Q(a)} \text{notI} \\ \frac{\forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \neg \exists a \bullet P(a) \wedge Q(a)}{\vdash (\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))} \text{impI} \end{array}$$

$x \notin \text{fv}(\forall a \bullet P(a) \Rightarrow \neg Q(a), \text{false})$



## Example: Combining Quantifiers

►  $(\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))$

► Proof

$$\begin{array}{c} \frac{\frac{\frac{\frac{P(x) \Rightarrow \neg Q(x), P(x), Q(x) \vdash \text{false}}{\forall a \bullet P(a) \Rightarrow \neg Q(a), P(x), Q(x) \vdash \text{false}} \text{allE}}{P(x), Q(x), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}} \text{rotate} \times 2}{P(x) \wedge Q(x), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}} \text{conjE}}{\exists a \bullet P(a) \wedge Q(a), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}} \text{exE}} \\ \frac{\exists a \bullet P(a) \wedge Q(a), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}}{\forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \neg \exists a \bullet P(a) \wedge Q(a)} \text{notI} \\ \frac{\forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \neg \exists a \bullet P(a) \wedge Q(a)}{\vdash (\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))} \text{impI} \end{array}$$

$x \notin \text{fv}(\forall a \bullet P(a) \Rightarrow \neg Q(a), \text{false})$

## Natural Deduction Rules

- $$\begin{array}{c}
\frac{}{P(x) \Rightarrow \neg Q(x), P(x), Q(x) \vdash \text{false}} \text{impE} \\
\frac{}{\forall a \bullet P(a) \Rightarrow \neg Q(a), P(x), Q(x) \vdash \text{false}} \text{allE} \\
\frac{}{P(x), Q(x), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}} \text{rotate} \times 2 \\
\frac{}{P(x) \wedge Q(x), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}} \text{conjE} \\
\frac{x \notin \text{fv}(\forall a \bullet P(a) \Rightarrow \neg Q(a), \text{false}) \quad P(x) \wedge Q(x), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}}{\exists a \bullet P(a) \wedge Q(a), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}} \text{exE} \\
\frac{}{\forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \neg \exists a \bullet P(a) \wedge Q(a)} \text{notI} \\
\frac{}{\vdash (\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))} \text{impI}
\end{array}$$

# Example: Combining Quantifiers

►  $(\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))$

► Proof

$$\begin{array}{c}
 \frac{P(x), Q(x) \vdash P(x) \quad \neg Q(x), P(x), Q(x) \vdash \text{false}}{P(x) \Rightarrow \neg Q(x), P(x), Q(x) \vdash \text{false}} \text{impE} \\
 \frac{P(x) \Rightarrow \neg Q(x), P(x), Q(x) \vdash \text{false}}{\forall a \bullet P(a) \Rightarrow \neg Q(a), P(x), Q(x) \vdash \text{false}} \text{allE} \\
 \frac{\forall a \bullet P(a) \Rightarrow \neg Q(a), P(x), Q(x) \vdash \text{false}}{P(x), Q(x), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}} \text{rotate} \times 2 \\
 \frac{P(x), Q(x), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}}{P(x) \wedge Q(x), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}} \text{conjE} \\
 \frac{P(x) \wedge Q(x), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}}{\exists a \bullet P(a) \wedge Q(a), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}} \text{exE} \\
 \frac{\exists a \bullet P(a) \wedge Q(a), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}}{\forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \neg \exists a \bullet P(a) \wedge Q(a)} \text{notI} \\
 \frac{\forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \neg \exists a \bullet P(a) \wedge Q(a)}{\vdash (\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))} \text{impI}
 \end{array}$$

$x \notin \text{fv}(\forall a \bullet P(a) \Rightarrow \neg Q(a), \text{false})$

# Example: Combining Quantifiers

►  $(\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))$

► Proof

$$\begin{array}{c}
 \frac{P(x), Q(x) \vdash P(x) \quad \overline{\neg Q(x), P(x), Q(x) \vdash \text{false}}}{P(x) \Rightarrow \neg Q(x), P(x), Q(x) \vdash \text{false}} \text{impE} \\
 \frac{\overline{P(x) \Rightarrow \neg Q(x), P(x), Q(x) \vdash \text{false}}}{\forall a \bullet P(a) \Rightarrow \neg Q(a), P(x), Q(x) \vdash \text{false}} \text{allE} \\
 \frac{\overline{\forall a \bullet P(a) \Rightarrow \neg Q(a), P(x), Q(x) \vdash \text{false}}}{P(x), Q(x), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}} \text{rotate} \times 2 \\
 \frac{\overline{P(x), Q(x), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}}}{P(x) \wedge Q(x), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}} \text{conjE} \\
 \frac{\overline{P(x) \wedge Q(x), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}}}{\exists a \bullet P(a) \wedge Q(a), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}} \text{exE} \\
 \frac{\overline{\exists a \bullet P(a) \wedge Q(a), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}}}{\forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \neg \exists a \bullet P(a) \wedge Q(a)} \text{notI} \\
 \frac{\overline{\forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \neg \exists a \bullet P(a) \wedge Q(a)}}{\vdash (\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))} \text{impI}
 \end{array}$$

$x \notin \text{fv}(\forall a \bullet P(a) \Rightarrow \neg Q(a), \text{false})$

## Natural Deduction Rules

- $$\begin{array}{c}
\frac{P(x), Q(x) \vdash P(x) \quad \neg Q(x), P(x), Q(x) \vdash \text{false}}{P(x) \Rightarrow \neg Q(x), P(x), Q(x) \vdash \text{false}} \text{notE} \\
\frac{P(x) \Rightarrow \neg Q(x), P(x), Q(x) \vdash \text{false}}{\forall a \bullet P(a) \Rightarrow \neg Q(a), P(x), Q(x) \vdash \text{false}} \text{impE} \\
\frac{\forall a \bullet P(a) \Rightarrow \neg Q(a), P(x), Q(x) \vdash \text{false}}{P(x), Q(x), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}} \text{allE} \\
\frac{P(x), Q(x), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}}{P(x) \wedge Q(x), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}} \text{rotate} \times 2 \\
\frac{P(x) \wedge Q(x), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}}{\exists a \bullet P(a) \wedge Q(a), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}} \text{conjE} \\
\frac{\exists a \bullet P(a) \wedge Q(a), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}}{\forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \neg \exists a \bullet P(a) \wedge Q(a)} \text{exE} \\
\frac{\forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \neg \exists a \bullet P(a) \wedge Q(a)}{\vdash (\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))} \text{notI} \\
\text{impI}
\end{array}$$

# Example: Combining Quantifiers

►  $(\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))$

► Proof

$$\begin{array}{c}
 \frac{P(x), Q(x) \vdash Q(x)}{P(x), Q(x) \vdash P(x) \quad \neg Q(x), P(x), Q(x) \vdash \text{false}} \text{notE} \\
 \hline
 \frac{P(x), Q(x) \vdash P(x) \quad \neg Q(x), P(x), Q(x) \vdash \text{false}}{P(x) \Rightarrow \neg Q(x), P(x), Q(x) \vdash \text{false}} \text{impE} \\
 \hline
 \frac{P(x) \Rightarrow \neg Q(x), P(x), Q(x) \vdash \text{false}}{\forall a \bullet P(a) \Rightarrow \neg Q(a), P(x), Q(x) \vdash \text{false}} \text{allE} \\
 \hline
 \frac{\forall a \bullet P(a) \Rightarrow \neg Q(a), P(x), Q(x) \vdash \text{false}}{P(x), Q(x), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}} \text{rotate} \times 2 \\
 \hline
 \frac{P(x), Q(x), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}}{P(x) \wedge Q(x), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}} \text{conjE} \\
 \hline
 \frac{P(x) \wedge Q(x), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}}{\exists a \bullet P(a) \wedge Q(a), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}} \text{exE} \\
 \hline
 \frac{\exists a \bullet P(a) \wedge Q(a), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}}{\forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \neg \exists a \bullet P(a) \wedge Q(a)} \text{notI} \\
 \hline
 \frac{\forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \neg \exists a \bullet P(a) \wedge Q(a)}{\vdash (\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))} \text{impI}
 \end{array}$$

# Example: Combining Quantifiers

►  $(\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))$

► Proof

$$\begin{array}{c}
 \frac{P(x), Q(x) \vdash Q(x)}{\neg Q(x), P(x), Q(x) \vdash \text{false}} \text{notE} \\
 \frac{P(x), Q(x) \vdash P(x) \quad \neg Q(x), P(x), Q(x) \vdash \text{false}}{P(x) \Rightarrow \neg Q(x), P(x), Q(x) \vdash \text{false}} \text{impE} \\
 \frac{P(x) \Rightarrow \neg Q(x), P(x), Q(x) \vdash \text{false}}{\forall a \bullet P(a) \Rightarrow \neg Q(a), P(x), Q(x) \vdash \text{false}} \text{allE} \\
 \frac{\forall a \bullet P(a) \Rightarrow \neg Q(a), P(x), Q(x) \vdash \text{false}}{P(x), Q(x), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}} \text{rotate} \times 2 \\
 \frac{P(x), Q(x), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}}{P(x) \wedge Q(x), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}} \text{conjE} \\
 \frac{P(x) \wedge Q(x), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}}{\exists a \bullet P(a) \wedge Q(a), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}} \text{exE} \\
 \frac{\exists a \bullet P(a) \wedge Q(a), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}}{\forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \neg \exists a \bullet P(a) \wedge Q(a)} \text{notI} \\
 \frac{\forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \neg \exists a \bullet P(a) \wedge Q(a)}{\vdash (\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))} \text{impI}
 \end{array}$$

$x \notin \text{fv}(\forall a \bullet P(a) \Rightarrow \neg Q(a), \text{false})$

## Natural Deduction Rules

- $$x \notin fv(\forall a \bullet P(a) \Rightarrow \neg Q(a), false)$$

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# Example: Combining Quantifiers

►  $(\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))$

► Proof

$$\begin{array}{c}
 \frac{Q(x), P(x) \vdash Q(x)}{P(x), Q(x) \vdash Q(x)} \text{rotate} \\
 \frac{P(x), Q(x) \vdash P(x) \quad \neg Q(x), P(x), Q(x) \vdash \text{false}}{P(x), Q(x) \vdash \text{false}} \text{notE} \\
 \frac{P(x), Q(x) \vdash \text{false}}{P(x) \Rightarrow \neg Q(x), P(x), Q(x) \vdash \text{false}} \text{impE} \\
 \frac{P(x) \Rightarrow \neg Q(x), P(x), Q(x) \vdash \text{false}}{\forall a \bullet P(a) \Rightarrow \neg Q(a), P(x), Q(x) \vdash \text{false}} \text{allE} \\
 \frac{\forall a \bullet P(a) \Rightarrow \neg Q(a), P(x), Q(x) \vdash \text{false}}{P(x), Q(x), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}} \text{rotate} \times 2 \\
 \frac{P(x), Q(x), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}}{P(x) \wedge Q(x), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}} \text{conjE} \\
 \frac{P(x) \wedge Q(x), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}}{\exists a \bullet P(a) \wedge Q(a), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}} \text{exE} \\
 \frac{\exists a \bullet P(a) \wedge Q(a), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}}{\forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \neg \exists a \bullet P(a) \wedge Q(a)} \text{notI} \\
 \frac{\forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \neg \exists a \bullet P(a) \wedge Q(a)}{\vdash (\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))} \text{impI}
 \end{array}$$

$x \notin \text{fv}(\forall a \bullet P(a) \Rightarrow \neg Q(a), \text{false})$

# Example: Combining Quantifiers

►  $(\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))$

► Proof

$$\begin{array}{c}
 \frac{}{Q(x), P(x) \vdash Q(x)} \text{rotate} \\
 \frac{}{P(x), Q(x) \vdash Q(x)} \text{notE} \\
 \frac{P(x), Q(x) \vdash P(x) \quad \neg Q(x), P(x), Q(x) \vdash \text{false}}{P(x), Q(x) \vdash P(x) \Rightarrow \neg Q(x)} \text{impE} \\
 \frac{P(x) \Rightarrow \neg Q(x), P(x), Q(x) \vdash \text{false}}{\forall a \bullet P(a) \Rightarrow \neg Q(a), P(x), Q(x) \vdash \text{false}} \text{allE} \\
 \frac{\forall a \bullet P(a) \Rightarrow \neg Q(a), P(x), Q(x) \vdash \text{false}}{P(x), Q(x), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}} \text{rotate} \times 2 \\
 \frac{P(x), Q(x), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}}{P(x) \wedge Q(x), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}} \text{conjE} \\
 \frac{P(x) \wedge Q(x), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}}{\exists a \bullet P(a) \wedge Q(a), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}} \text{exE} \\
 \frac{\exists a \bullet P(a) \wedge Q(a), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}}{\forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \neg \exists a \bullet P(a) \wedge Q(a)} \text{notI} \\
 \frac{\forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \neg \exists a \bullet P(a) \wedge Q(a)}{\vdash (\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))} \text{impI}
 \end{array}$$

$x \notin \text{fv}(\forall a \bullet P(a) \Rightarrow \neg Q(a), \text{false})$

# Example: Combining Quantifiers

►  $(\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))$

► Proof

$$\begin{array}{c}
 \frac{}{Q(x), P(x) \vdash Q(x)} \text{asm} \\
 \frac{}{P(x), Q(x) \vdash Q(x)} \text{rotate} \\
 \frac{P(x), Q(x) \vdash P(x) \quad \neg Q(x), P(x), Q(x) \vdash \text{false}}{P(x), Q(x) \vdash P(x) \Rightarrow \neg Q(x)} \text{notE} \\
 \frac{}{P(x) \Rightarrow \neg Q(x), P(x), Q(x) \vdash \text{false}} \text{impE} \\
 \frac{}{\forall a \bullet P(a) \Rightarrow \neg Q(a), P(x), Q(x) \vdash \text{false}} \text{allE} \\
 \frac{}{P(x), Q(x), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}} \text{rotate} \times 2 \\
 \frac{}{P(x) \wedge Q(x), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}} \text{conjE} \\
 \frac{}{\exists a \bullet P(a) \wedge Q(a), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}} \text{exE} \\
 \frac{}{\forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \neg \exists a \bullet P(a) \wedge Q(a)} \text{notI} \\
 \frac{}{\vdash (\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))} \text{impI}
 \end{array}$$

$x \notin \text{fv}(\forall a \bullet P(a) \Rightarrow \neg Q(a), \text{false})$

# Example: Combining Quantifiers

►  $(\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))$

► Proof

$$\begin{array}{c}
 \frac{}{Q(x), P(x) \vdash Q(x)} \text{asm} \\
 \frac{}{P(x), Q(x) \vdash Q(x)} \text{rotate} \\
 \frac{P(x), Q(x) \vdash P(x) \quad \neg Q(x), P(x), Q(x) \vdash \text{false}}{P(x), Q(x) \vdash P(x) \Rightarrow \neg Q(x)} \text{notE} \\
 \frac{}{P(x) \Rightarrow \neg Q(x), P(x), Q(x) \vdash \text{false}} \text{impE} \\
 \frac{}{\forall a \bullet P(a) \Rightarrow \neg Q(a), P(x), Q(x) \vdash \text{false}} \text{allE} \\
 \frac{}{P(x), Q(x), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}} \text{rotate} \times 2 \\
 \frac{}{P(x) \wedge Q(x), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}} \text{conjE} \\
 \frac{}{\exists a \bullet P(a) \wedge Q(a), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}} \text{exE} \\
 \frac{}{\forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \neg \exists a \bullet P(a) \wedge Q(a)} \text{notI} \\
 \frac{}{\vdash (\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))} \text{impI}
 \end{array}$$

$x \notin \text{fv}(\forall a \bullet P(a) \Rightarrow \neg Q(a), \text{false})$

# Example: Combining Quantifiers

►  $(\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))$

► Proof

$$\begin{array}{c}
 \frac{}{\neg} \text{asm} \\
 \frac{}{Q(x), P(x) \vdash Q(x)} \text{rotate} \\
 \frac{}{P(x), Q(x) \vdash Q(x)} \text{notE} \\
 \frac{P(x), Q(x) \vdash P(x)}{\neg Q(x), P(x), Q(x) \vdash \text{false}} \text{impE} \\
 \frac{P(x) \Rightarrow \neg Q(x), P(x), Q(x) \vdash \text{false}}{\forall a \bullet P(a) \Rightarrow \neg Q(a), P(x), Q(x) \vdash \text{false}} \text{allE} \\
 \frac{\forall a \bullet P(a) \Rightarrow \neg Q(a), P(x), Q(x) \vdash \text{false}}{P(x), Q(x), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}} \text{rotate} \times 2 \\
 \frac{P(x), Q(x), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}}{P(x) \wedge Q(x), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}} \text{conjE} \\
 \frac{P(x) \wedge Q(x), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}}{\exists a \bullet P(a) \wedge Q(a), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}} \text{exE} \\
 \frac{\exists a \bullet P(a) \wedge Q(a), \forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \text{false}}{\forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \neg \exists a \bullet P(a) \wedge Q(a)} \text{notI} \\
 \frac{\forall a \bullet P(a) \Rightarrow \neg Q(a) \vdash \neg \exists a \bullet P(a) \wedge Q(a)}{\vdash (\forall a \bullet P(a) \Rightarrow \neg Q(a)) \Rightarrow (\neg \exists a \bullet P(a) \wedge Q(a))} \text{impI}
 \end{array}$$

$x \notin \text{fv}(\forall a \bullet P(a) \Rightarrow \neg Q(a), \text{false})$

## Natural Deduction Rules

- $$x \notin fv(\forall a \bullet P(a) \Rightarrow \neg Q(a), false)$$

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## Natural Deduction Rules

- $$x \notin fv(\forall a \bullet P(a) \Rightarrow \neg Q(a), false)$$

20/21

# Summary

- ▶ Natural deduction rules for predicate calculus quantifiers.

Universal Introduction    safe

Universal Elimination    unsafe

Existential Introduction    unsafe

Existential Elimination    safe



# Summary

- ▶ Natural deduction rules for predicate calculus quantifiers.

Universal Introduction    safe

Universal Elimination    unsafe

Existential Introduction    unsafe

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