Natural Deduction
Propositional Calculus

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PROF

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Overview

Outline

Natural Deduction Rules

$$\frac{-}{P, \Gamma \vdash P}\ \texttt{asm} \qquad \frac{\Gamma \vdash Q}{P, \Gamma \vdash Q}\ \texttt{thin} \qquad \frac{\Gamma, P \vdash Q}{P, \Gamma \vdash Q}\ \texttt{rotate} \qquad \frac{-}{\Gamma \vdash t = t}\ \texttt{refl}$$

$$\frac{s = t, \Gamma \vdash P(t)}{s = t, \Gamma \vdash P(s)}\ \texttt{subst} \qquad \frac{\Gamma \vdash f(x) = g(x) \qquad x \notin fv(\Gamma)}{\Gamma \vdash f = g}\ \texttt{ext}$$

$$\frac{-}{\Gamma \vdash true}\ \texttt{TrueI} \qquad \frac{-}{false, \Gamma \vdash P}\ \texttt{FalseE}$$

$$\frac{\Gamma \vdash P \qquad \Gamma \vdash Q}{\Gamma \vdash P \wedge Q}\ \texttt{conjI} \qquad \frac{P, Q, \Gamma \vdash R}{P \wedge Q, \Gamma \vdash R}\ \texttt{conjE}$$

$$\frac{\Gamma \vdash P}{\Gamma \vdash P \vee Q}\ \texttt{disjI1} \qquad \frac{\Gamma \vdash Q}{\Gamma \vdash P \vee Q}\ \texttt{disjI2} \qquad \frac{P, \Gamma \vdash R \qquad Q, \Gamma \vdash R}{P \vee Q, \Gamma \vdash R}\ \texttt{disjE}$$

$$\frac{P, \Gamma \vdash Q}{\Gamma \vdash P \Rightarrow Q}\ \texttt{impI} \qquad \frac{\Gamma \vdash P \qquad Q, \Gamma \vdash R}{P \Rightarrow Q, \Gamma \vdash R}\ \texttt{impE}$$

$$\frac{\Gamma \vdash P \Rightarrow Q \qquad \Gamma \vdash Q \Rightarrow P}{\Gamma \vdash P \Leftrightarrow Q}\ \texttt{iffI} \qquad \frac{P \Rightarrow Q, Q \Rightarrow P, \Gamma \vdash R}{P \Leftrightarrow Q, \Gamma \vdash R}\ \texttt{iffE}$$

$$\frac{P, \Gamma \vdash false}{\Gamma \vdash \neg P}\ \texttt{notI} \qquad \frac{\Gamma \vdash P}{\neg P, \Gamma \vdash R}\ \texttt{notE} \qquad \frac{\neg P, \Gamma \vdash false}{\Gamma \vdash P}\ \texttt{ccontr} \qquad \frac{\neg Q, \Gamma \vdash P}{\Gamma \vdash P \vee Q}\ \texttt{disjCI}$$

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Gentzen Sequent Calculus

- Style of formal logical argumentation on conditional tautologies: sequents.
- Natural deduction: every sequent has exactly one conclusion.
- $A_1, A_2, \ldots, A_n \vdash P$, where the A_i's and P are propositions and $n \in \mathbb{N}$.
- Each sequent is inferred from sequents on earlier lines in a formal argument.
- Each step appeals to a precise rule of inference.
- Theorems: formulas P such that $\vdash P$ is the conclusion of a valid proof.
- Convention: Γ is a list of hypotheses. Example: $\Gamma \vdash P$.

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Inference rules

- Inferences on sequents use a system of natural deduction rules.
- A collection of sound inference rules.

  Schematic example:

  If all of the premises S_i hold, then the goal T must hold: $\dfrac{S_1 \quad \cdots \quad S_n}{T}$

- Example: $\dfrac{\Gamma \vdash P \qquad \Gamma \vdash Q}{\Gamma \vdash P \wedge Q}\,\texttt{conjI}$
- Interpretation: If we have a proof of the validity of $\Gamma \vdash P$ and of $\Gamma \vdash Q$, then we can infer that $\Gamma \vdash P \wedge Q$ is also valid.
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Naming

- Most of our rules are associated with a particular connective.
- An introduction rule has the connective in the conclusion of its goal.
- An elimination rule has the connective in the hypotheses of its goal.
- Example: conjunction.

Conjunction introduction: $$\frac{\Gamma \vdash P \qquad \Gamma \vdash Q}{\Gamma \vdash P \land Q}\,\texttt{conjI}$$

Conjunction elimination: $$\frac{P, Q, \Gamma \vdash R}{P \land Q, \Gamma \vdash R}\,\texttt{conjE}$$

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Proof Trees

- Shows how a conclusion can be reached from a set of premises.
- Formal proof: every step justified from basic rules.
- Example:
- `asm`, `conjI`, `conjE`, `rotate`, and `impI` are all valid rule names properly applied.
- So it's a formal proof of the theorem $P \wedge Q \Rightarrow Q \wedge P$.
- Note that complete proof trees have empty premises (leaves).

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- Shows how a conclusion can be reached from a set of premises.
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- asm, conjI, conjE, rotate, and impI are all valid rule names properly applied.
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The Rules of Inference

- P, Q, R range over propositions.
- x, y, z range over fixed variables.
- s, t range over terms.
- f, g range over total functions.
- A, B range over sets.
- Variable Γ is a proof context, consisting of a sequence of propositions, $P, Q, \cdots, R$.
- Meta-logical function fv returns the set of variables free in a term or context.

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Outline

Inference Rule: Assumption

- The simplest proof rule proves a goal from the hypothesis: $\dfrac{-}{P, \Gamma \vdash P}\,\texttt{asm}$
- The goal $P, \Gamma \vdash P$ is replaced by the empty goal and the proof is finished.
- Example

  Suppose we want to prove an arbitrary proposition P.

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  Our proof is immediate! But what have we proved?
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Manipulating Hypotheses: Thinning

- Structural Rule.
- Unnecessary hypotheses can be removed in a proof: $\dfrac{\Gamma \vdash Q}{P, \Gamma \vdash Q}\ \mathtt{thin}$
- Example:

  Suppose we know some facts about x: $0 \leq x \leq 10$ and $x \in \mathbb{N}$.

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  We have more hypotheses than we need:
- The proof will now follow from the fact that $\mathbb{N} \subseteq \mathbb{Z}$.

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- The order of hypotheses can be changed: $\dfrac{\Gamma, P \vdash Q}{P, \Gamma \vdash Q}$ `rotate`
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  Suppose our facts in the last example were presented in a more logical order:

  Some facts about x: $x \in \mathbb{N}$ and $0 \leq x \leq 10$.

- We can now proceed as we did before.
- We often need to reorder hypotheses to match inference rules.

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- The order of hypotheses can be changed: $\dfrac{\Gamma, P \vdash Q}{P, \Gamma \vdash Q}$ `rotate`
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  Some facts about x: $x \in \mathbb{N}$ and $0 \leq x \leq 10$.

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Truth Constants

- The truth constants are *true* and *false*.
- They can both be used to complete a proof branch.
- If our goal has the conclusion *true*, then there is nothing left to prove:

$$\frac{-}{\Gamma \vdash \mathit{true}}\ \texttt{TrueI}$$

- We are trying to prove a trivial tautology, which is true for any hypothesis.
- If our goal has the hypothesis *false*, then there is nothing left to prove:

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Outline

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- As we have already seen, conjunction has two rules of reasoning.
- Conjunction introduction: conjuncts can be proved separately:

$$\frac{\Gamma \vdash P \qquad \Gamma \vdash Q}{\Gamma \vdash P \wedge Q}\ \texttt{conjI}$$

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$$\dfrac{\dfrac{\overline{Q \wedge R, P \vdash (P \wedge Q) \wedge R}}{P, Q \wedge R \vdash (P \wedge Q) \wedge R}\,\texttt{rotate}}{P \wedge (Q \wedge R) \vdash (P \wedge Q) \wedge R}\,\texttt{conjE}$$

Example: Conjunction is Associative

$$\dfrac{\dfrac{\dfrac{}{Q \wedge R, P \vdash (P \wedge Q) \wedge R}\;\texttt{conjE}}{P, Q \wedge R \vdash (P \wedge Q) \wedge R}\;\texttt{rotate}}{P \wedge (Q \wedge R) \vdash (P \wedge Q) \wedge R}\;\texttt{conjE}$$

Example: Conjunction is Associative

$$\dfrac{\dfrac{\dfrac{Q, R, P \vdash (P \wedge Q) \wedge R}{Q \wedge R, P \vdash (P \wedge Q) \wedge R}\,\texttt{conjE}}{P, Q \wedge R \vdash (P \wedge Q) \wedge R}\,\texttt{rotate}}{P \wedge (Q \wedge R) \vdash (P \wedge Q) \wedge R}\,\texttt{conjE}$$

Example: Conjunction is Associative

$$\dfrac{\dfrac{\dfrac{\dfrac{}{Q, R, P \vdash (P \wedge Q) \wedge R}}{Q \wedge R, P \vdash (P \wedge Q) \wedge R}\ \texttt{conjE}}{P, Q \wedge R \vdash (P \wedge Q) \wedge R}\ \texttt{rotate}}{P \wedge (Q \wedge R) \vdash (P \wedge Q) \wedge R}\ \texttt{conjE}$$

Example: Conjunction is Associative

$$\dfrac{\dfrac{\dfrac{\dfrac{}{Q, R, P \vdash (P \wedge Q) \wedge R}\ \texttt{conjI}}{Q \wedge R, P \vdash (P \wedge Q) \wedge R}\ \texttt{conjE}}{P, Q \wedge R \vdash (P \wedge Q) \wedge R}\ \texttt{rotate}}{P \wedge (Q \wedge R) \vdash (P \wedge Q) \wedge R}\ \texttt{conjE}$$

Example: Conjunction is Associative

$$\dfrac{\dfrac{\dfrac{\dfrac{Q, R, P \vdash P \wedge Q \qquad\qquad Q, R, P \vdash R}{Q, R, P \vdash (P \wedge Q) \wedge R}\ \texttt{conjI}}{Q \wedge R, P \vdash (P \wedge Q) \wedge R}\ \texttt{conjE}}{P, Q \wedge R \vdash (P \wedge Q) \wedge R}\ \texttt{rotate}}{P \wedge (Q \wedge R) \vdash (P \wedge Q) \wedge R}\ \texttt{conjE}$$

Example: Conjunction is Associative

$$
\dfrac{\dfrac{\dfrac{\dfrac{Q, R, P \vdash P \wedge Q \qquad \dfrac{}{Q, R, P \vdash R}}{Q, R, P \vdash (P \wedge Q) \wedge R}\,\texttt{conjI}}{Q \wedge R, P \vdash (P \wedge Q) \wedge R}\,\texttt{conjE}}{P, Q \wedge R \vdash (P \wedge Q) \wedge R}\,\texttt{rotate}}{P \wedge (Q \wedge R) \vdash (P \wedge Q) \wedge R}\,\texttt{conjE}
$$

Example: Conjunction is Associative

$$
\dfrac{\dfrac{\dfrac{Q, R, P \vdash P \wedge Q \qquad \dfrac{}{Q, R, P \vdash R}\,\texttt{rotate}}{Q, R, P \vdash (P \wedge Q) \wedge R}\,\texttt{conjI}}{\dfrac{Q \wedge R, P \vdash (P \wedge Q) \wedge R}{P, Q \wedge R \vdash (P \wedge Q) \wedge R}\,\texttt{rotate}}\,\texttt{conjE}}{P \wedge (Q \wedge R) \vdash (P \wedge Q) \wedge R}\,\texttt{conjE}
$$

Example: Conjunction is Associative

$$\dfrac{\dfrac{\dfrac{Q, R, P \vdash P \wedge Q \qquad \dfrac{R, P, Q \vdash R}{Q, R, P \vdash R}\ \texttt{rotate}}{Q, R, P \vdash (P \wedge Q) \wedge R}\ \texttt{conjI}}{Q \wedge R, P \vdash (P \wedge Q) \wedge R}\ \texttt{conjE}}{\dfrac{P, Q \wedge R \vdash (P \wedge Q) \wedge R}{P \wedge (Q \wedge R) \vdash (P \wedge Q) \wedge R}\ \texttt{conjE}}\ \texttt{rotate}$$

Example: Conjunction is Associative

$$
\dfrac{\dfrac{\dfrac{\dfrac{Q, R, P \vdash P \wedge Q \qquad \dfrac{\dfrac{}{R, P, Q \vdash R}}{Q, R, P \vdash R}\,\texttt{rotate}}{Q, R, P \vdash (P \wedge Q) \wedge R}\,\texttt{conjI}}{Q \wedge R, P \vdash (P \wedge Q) \wedge R}\,\texttt{conjE}}{P, Q \wedge R \vdash (P \wedge Q) \wedge R}\,\texttt{rotate}}{P \wedge (Q \wedge R) \vdash (P \wedge Q) \wedge R}\,\texttt{conjE}
$$

Example: Conjunction is Associative

$$
\dfrac{\dfrac{Q, R, P \vdash P \wedge Q \qquad \dfrac{\dfrac{}{R, P, Q \vdash R}\,\texttt{asm}}{Q, R, P \vdash R}\,\texttt{rotate}}{\dfrac{Q, R, P \vdash (P \wedge Q) \wedge R}{\dfrac{Q \wedge R, P \vdash (P \wedge Q) \wedge R}{P, Q \wedge R \vdash (P \wedge Q) \wedge R}\,\texttt{rotate}}\,\texttt{conjE}}\,\texttt{conjI}}{P \wedge (Q \wedge R) \vdash (P \wedge Q) \wedge R}\,\texttt{conjE}
$$

Example: Conjunction is Associative

Natural Deduction Rules

$$
\cfrac{\cfrac{\cfrac{\cfrac{Q, R, P \vdash P \wedge Q \qquad \cfrac{\cfrac{-}{R, P, Q \vdash R}\;\texttt{asm}}{Q, R, P \vdash R}\;\texttt{rotate}}{Q, R, P \vdash (P \wedge Q) \wedge R}\;\texttt{conjI}}{Q \wedge R, P \vdash (P \wedge Q) \wedge R}\;\texttt{conjE}}{P, Q \wedge R \vdash (P \wedge Q) \wedge R}\;\texttt{rotate}}{P \wedge (Q \wedge R) \vdash (P \wedge Q) \wedge R}\;\texttt{conjE}
$$

Example: Conjunction is Associative

$$
\dfrac{\dfrac{}{Q, R, P \vdash P \wedge Q} \qquad \dfrac{\dfrac{\dfrac{-}{R, P, Q \vdash R}\ \texttt{asm}}{Q, R, P \vdash R}\ \texttt{rotate}}{}}{\dfrac{Q, R, P \vdash (P \wedge Q) \wedge R}{\dfrac{Q \wedge R, P \vdash (P \wedge Q) \wedge R}{\dfrac{P, Q \wedge R \vdash (P \wedge Q) \wedge R}{P \wedge (Q \wedge R) \vdash (P \wedge Q) \wedge R}\ \texttt{conjE}}\ \texttt{rotate}}\ \texttt{conjE}}\ \texttt{conjI}
$$

Example: Conjunction is Associative

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{Q, R, P \vdash P \wedge Q}\;\texttt{conjI} \qquad \dfrac{\dfrac{-}{R, P, Q \vdash R}\;\texttt{asm}}{Q, R, P \vdash R}\;\texttt{rotate}}{Q, R, P \vdash (P \wedge Q) \wedge R}\;\texttt{conjI}}{Q \wedge R, P \vdash (P \wedge Q) \wedge R}\;\texttt{conjE}}{P, Q \wedge R \vdash (P \wedge Q) \wedge R}\;\texttt{rotate}}{P \wedge (Q \wedge R) \vdash (P \wedge Q) \wedge R}\;\texttt{conjE}
$$

Example: Conjunction is Associative

$$
\dfrac{\dfrac{Q, R, P \vdash P \qquad Q, R, P \vdash Q}{Q, R, P \vdash P \land Q}\,\texttt{conjI} \qquad \dfrac{\dfrac{\dfrac{-}{R, P, Q \vdash R}\,\texttt{asm}}{Q, R, P \vdash R}\,\texttt{rotate}}{}}{\dfrac{\dfrac{\dfrac{Q, R, P \vdash (P \land Q) \land R}{Q \land R, P \vdash (P \land Q) \land R}\,\texttt{conjE}}{P, Q \land R \vdash (P \land Q) \land R}\,\texttt{rotate}}{P \land (Q \land R) \vdash (P \land Q) \land R}\,\texttt{conjE}}\,\texttt{conjI}
$$

Example: Conjunction is Associative

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{Q, R, P \vdash P \qquad \dfrac{}{Q, R, P \vdash Q}}{Q, R, P \vdash P \wedge Q}\,\texttt{conjI} \qquad \dfrac{\dfrac{-}{R, P, Q \vdash R}\,\texttt{asm}}{Q, R, P \vdash R}\,\texttt{rotate}}{Q, R, P \vdash (P \wedge Q) \wedge R}\,\texttt{conjI}}{Q \wedge R, P \vdash (P \wedge Q) \wedge R}\,\texttt{conjE}}{P, Q \wedge R \vdash (P \wedge Q) \wedge R}\,\texttt{rotate}}{P \wedge (Q \wedge R) \vdash (P \wedge Q) \wedge R}\,\texttt{conjE}
$$

Example: Conjunction is Associative

$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{Q, R, P \vdash P \qquad \cfrac{}{Q, R, P \vdash Q}\;\texttt{asm}}{Q, R, P \vdash P \wedge Q}\;\texttt{conjI} \qquad \cfrac{\cfrac{}{R, P, Q \vdash R}\;\texttt{asm}}{Q, R, P \vdash R}\;\texttt{rotate}}{Q, R, P \vdash (P \wedge Q) \wedge R}\;\texttt{conjI}}{Q \wedge R, P \vdash (P \wedge Q) \wedge R}\;\texttt{conjE}}{P, Q \wedge R \vdash (P \wedge Q) \wedge R}\;\texttt{rotate}}{P \wedge (Q \wedge R) \vdash (P \wedge Q) \wedge R}\;\texttt{conjE}
$$

Example: Conjunction is Associative

$$
\cfrac{\cfrac{\cfrac{\cfrac{Q, R, P \vdash P \qquad \cfrac{-}{Q, R, P \vdash Q}\,\texttt{asm}}{Q, R, P \vdash P \wedge Q}\,\texttt{conjI} \qquad \cfrac{\cfrac{-}{R, P, Q \vdash R}\,\texttt{asm}}{Q, R, P \vdash R}\,\texttt{rotate}}{Q, R, P \vdash (P \wedge Q) \wedge R}\,\texttt{conjI}}{\cfrac{Q \wedge R, P \vdash (P \wedge Q) \wedge R}{P, Q \wedge R \vdash (P \wedge Q) \wedge R}\,\texttt{rotate}}\,\texttt{conjE}}{P \wedge (Q \wedge R) \vdash (P \wedge Q) \wedge R}\,\texttt{conjE}
$$

Example: Conjunction is Associative

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{Q, R, P \vdash P} \qquad \dfrac{-}{Q, R, P \vdash Q}\,\texttt{asm}}{Q, R, P \vdash P \wedge Q}\,\texttt{conjI} \qquad \dfrac{\dfrac{-}{R, P, Q \vdash R}\,\texttt{asm}}{Q, R, P \vdash R}\,\texttt{rotate}}{Q, R, P \vdash (P \wedge Q) \wedge R}\,\texttt{conjI}}{Q \wedge R, P \vdash (P \wedge Q) \wedge R}\,\texttt{conjE}}{\dfrac{P, Q \wedge R \vdash (P \wedge Q) \wedge R}{P \wedge (Q \wedge R) \vdash (P \wedge Q) \wedge R}\,\texttt{conjE}}\,\texttt{rotate}
$$

Example: Conjunction is Associative

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{Q, R, P \vdash P}\,\texttt{rotate} \qquad \dfrac{-}{Q, R, P \vdash Q}\,\texttt{asm}}{Q, R, P \vdash P \wedge Q}\,\texttt{conjI} \qquad \dfrac{\dfrac{-}{R, P, Q \vdash R}\,\texttt{asm}}{Q, R, P \vdash R}\,\texttt{rotate}}{Q, R, P \vdash (P \wedge Q) \wedge R}\,\texttt{conjI}}{Q \wedge R, P \vdash (P \wedge Q) \wedge R}\,\texttt{conjE}}{P, Q \wedge R \vdash (P \wedge Q) \wedge R}\,\texttt{rotate}}{P \wedge (Q \wedge R) \vdash (P \wedge Q) \wedge R}\,\texttt{conjE}$$

Example: Conjunction is Associative

$$
\begin{prooftree}
\AxiomC{$R, P, Q \vdash P$}
\RightLabel{\texttt{rotate}}
\UnaryInfC{$Q, R, P \vdash P$}
\AxiomC{$-$}
\RightLabel{\texttt{asm}}
\UnaryInfC{$Q, R, P \vdash Q$}
\RightLabel{\texttt{conjI}}
\BinaryInfC{$Q, R, P \vdash P \wedge Q$}
\AxiomC{$-$}
\RightLabel{\texttt{asm}}
\UnaryInfC{$R, P, Q \vdash R$}
\RightLabel{\texttt{rotate}}
\UnaryInfC{$Q, R, P \vdash R$}
\RightLabel{\texttt{conjI}}
\BinaryInfC{$Q, R, P \vdash (P \wedge Q) \wedge R$}
\RightLabel{\texttt{conjE}}
\UnaryInfC{$Q \wedge R, P \vdash (P \wedge Q) \wedge R$}
\RightLabel{\texttt{rotate}}
\UnaryInfC{$P, Q \wedge R \vdash (P \wedge Q) \wedge R$}
\RightLabel{\texttt{conjE}}
\UnaryInfC{$P \wedge (Q \wedge R) \vdash (P \wedge Q) \wedge R$}
\end{prooftree}
$$

Example: Conjunction is Associative

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{\dfrac{}{R, P, Q \vdash P}}{Q, R, P \vdash P}\;\mathtt{rotate}
      \qquad
      \dfrac{-}{Q, R, P \vdash Q}\;\mathtt{asm}
    }{Q, R, P \vdash P \wedge Q}\;\mathtt{conjI}
    \qquad
    \dfrac{\dfrac{-}{R, P, Q \vdash R}\;\mathtt{asm}}{Q, R, P \vdash R}\;\mathtt{rotate}
  }{
    \dfrac{
      \dfrac{Q, R, P \vdash (P \wedge Q) \wedge R}{Q \wedge R, P \vdash (P \wedge Q) \wedge R}\;\mathtt{conjE}
    }{P, Q \wedge R \vdash (P \wedge Q) \wedge R}\;\mathtt{rotate}
  }\;\mathtt{conjI}
}{P \wedge (Q \wedge R) \vdash (P \wedge Q) \wedge R}\;\mathtt{conjE}
$$

Example: Conjunction is Associative

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{R, P, Q \vdash P}\ \texttt{rotate}}{Q, R, P \vdash P}\ \texttt{rotate} \qquad \dfrac{-}{Q, R, P \vdash Q}\ \texttt{asm}}{Q, R, P \vdash P \wedge Q}\ \texttt{conjI} \qquad \dfrac{\dfrac{-}{R, P, Q \vdash R}\ \texttt{asm}}{Q, R, P \vdash R}\ \texttt{rotate}}{Q, R, P \vdash (P \wedge Q) \wedge R}\ \texttt{conjI}}{\dfrac{Q \wedge R, P \vdash (P \wedge Q) \wedge R}{P, Q \wedge R \vdash (P \wedge Q) \wedge R}\ \texttt{rotate}}\ \texttt{conjE}}{P \wedge (Q \wedge R) \vdash (P \wedge Q) \wedge R}\ \texttt{conjE}$$

Example: Conjunction is Associative

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{P, Q, R \vdash P}{R, P, Q \vdash P}\,\texttt{rotate}
      }{Q, R, P \vdash P}\,\texttt{rotate}
      \qquad
      \dfrac{-}{Q, R, P \vdash Q}\,\texttt{asm}
    }{Q, R, P \vdash P \wedge Q}\,\texttt{conjI}
    \qquad
    \dfrac{
      \dfrac{-}{R, P, Q \vdash R}\,\texttt{asm}
    }{Q, R, P \vdash R}\,\texttt{rotate}
  }{
    \dfrac{
      \dfrac{Q, R, P \vdash (P \wedge Q) \wedge R}{Q \wedge R, P \vdash (P \wedge Q) \wedge R}\,\texttt{conjE}
    }{P, Q \wedge R \vdash (P \wedge Q) \wedge R}\,\texttt{rotate}
  }\,\texttt{conjI}
}{P \wedge (Q \wedge R) \vdash (P \wedge Q) \wedge R}\,\texttt{conjE}
$$

Example: Conjunction is Associative

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P, Q, R \vdash P}}{R, P, Q \vdash P}\;\texttt{rotate}}{Q, R, P \vdash P}\;\texttt{rotate} \qquad \dfrac{-}{Q, R, P \vdash Q}\;\texttt{asm}}{Q, R, P \vdash P \wedge Q}\;\texttt{conjI} \qquad \dfrac{\dfrac{-}{R, P, Q \vdash R}\;\texttt{asm}}{Q, R, P \vdash R}\;\texttt{rotate}}{Q, R, P \vdash (P \wedge Q) \wedge R}\;\texttt{conjI}}{Q \wedge R, P \vdash (P \wedge Q) \wedge R}\;\texttt{conjE}}{P, Q \wedge R \vdash (P \wedge Q) \wedge R}\;\texttt{rotate}}{P \wedge (Q \wedge R) \vdash (P \wedge Q) \wedge R}\;\texttt{conjE}
$$

Example: Conjunction is Associative

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{\texttt{asm}}
\UnaryInfC{$P, Q, R \vdash P$}
\RightLabel{\texttt{rotate}}
\UnaryInfC{$R, P, Q \vdash P$}
\RightLabel{\texttt{rotate}}
\UnaryInfC{$Q, R, P \vdash P$}
\AxiomC{$-$}
\RightLabel{\texttt{asm}}
\UnaryInfC{$Q, R, P \vdash Q$}
\RightLabel{\texttt{conjI}}
\BinaryInfC{$Q, R, P \vdash P \wedge Q$}
\AxiomC{$-$}
\RightLabel{\texttt{asm}}
\UnaryInfC{$R, P, Q \vdash R$}
\RightLabel{\texttt{rotate}}
\UnaryInfC{$Q, R, P \vdash R$}
\RightLabel{\texttt{conjI}}
\BinaryInfC{$Q, R, P \vdash (P \wedge Q) \wedge R$}
\RightLabel{\texttt{conjE}}
\UnaryInfC{$Q \wedge R, P \vdash (P \wedge Q) \wedge R$}
\RightLabel{\texttt{rotate}}
\UnaryInfC{$P, Q \wedge R \vdash (P \wedge Q) \wedge R$}
\RightLabel{\texttt{conjE}}
\UnaryInfC{$P \wedge (Q \wedge R) \vdash (P \wedge Q) \wedge R$}
\end{prooftree}
$$

Example: Conjunction is Associative

$$
\begin{prooftree}
\AxiomC{$-$}
\RightLabel{$\texttt{asm}$}
\UnaryInfC{$P, Q, R \vdash P$}
\RightLabel{$\texttt{rotate}$}
\UnaryInfC{$R, P, Q \vdash P$}
\RightLabel{$\texttt{rotate}$}
\UnaryInfC{$Q, R, P \vdash P$}
\AxiomC{$-$}
\RightLabel{$\texttt{asm}$}
\UnaryInfC{$Q, R, P \vdash Q$}
\RightLabel{$\texttt{conjI}$}
\BinaryInfC{$Q, R, P \vdash P \wedge Q$}
\AxiomC{$-$}
\RightLabel{$\texttt{asm}$}
\UnaryInfC{$R, P, Q \vdash R$}
\RightLabel{$\texttt{rotate}$}
\UnaryInfC{$Q, R, P \vdash R$}
\RightLabel{$\texttt{conjI}$}
\BinaryInfC{$Q, R, P \vdash (P \wedge Q) \wedge R$}
\RightLabel{$\texttt{conjE}$}
\UnaryInfC{$Q \wedge R, P \vdash (P \wedge Q) \wedge R$}
\RightLabel{$\texttt{rotate}$}
\UnaryInfC{$P, Q \wedge R \vdash (P \wedge Q) \wedge R$}
\RightLabel{$\texttt{conjE}$}
\UnaryInfC{$P \wedge (Q \wedge R) \vdash (P \wedge Q) \wedge R$}
\end{prooftree}
$$

Example: Conjunction is Associative: Longer Proof

$$
\begin{prooftree}
\AxiomC{$-$}
\RightLabel{\texttt{asm}}
\UnaryInfC{$P, Q \wedge R \vdash P$}
\RightLabel{\texttt{conjE}}
\UnaryInfC{$P \wedge (Q \wedge R) \vdash P$}
\AxiomC{$-$}
\RightLabel{\texttt{asm}}
\UnaryInfC{$Q, R, P \vdash Q$}
\RightLabel{\texttt{conjE}}
\UnaryInfC{$Q \wedge R, P \vdash Q$}
\RightLabel{\texttt{rotate}}
\UnaryInfC{$P, Q \wedge R \vdash Q$}
\RightLabel{\texttt{conjE}}
\UnaryInfC{$P \wedge (Q \wedge R) \vdash Q$}
\RightLabel{\texttt{conjI}}
\BinaryInfC{$P \wedge (Q \wedge R) \vdash P \wedge Q$}
\AxiomC{$-$}
\RightLabel{\texttt{asm}}
\UnaryInfC{$R, P, Q \vdash R$}
\RightLabel{\texttt{rotate}}
\UnaryInfC{$Q, R, P \vdash R$}
\RightLabel{\texttt{conjE}}
\UnaryInfC{$Q \wedge R, P \vdash R$}
\RightLabel{\texttt{rotate}}
\UnaryInfC{$P, Q \wedge R \vdash R$}
\RightLabel{\texttt{conjE}}
\UnaryInfC{$P \wedge (Q \wedge R) \vdash R$}
\RightLabel{\texttt{conjI}}
\BinaryInfC{$P \wedge (Q \wedge R) \vdash (P \wedge Q) \wedge R$}
\end{prooftree}
$$

Example: Conjunction is Associative: Longer Proof

$$
\begin{prooftree}
\AxiomC{$-$}
\RightLabel{\texttt{asm}}
\UnaryInfC{$P, Q \wedge R \vdash P$}
\RightLabel{\texttt{conjE}}
\UnaryInfC{$P \wedge (Q \wedge R) \vdash P$}
\AxiomC{$-$}
\RightLabel{\texttt{asm}}
\UnaryInfC{$Q, R, P \vdash Q$}
\RightLabel{\texttt{conjE}}
\UnaryInfC{$Q \wedge R, P \vdash Q$}
\RightLabel{\texttt{rotate}}
\UnaryInfC{$P, Q \wedge R \vdash Q$}
\RightLabel{\texttt{conjE}}
\UnaryInfC{$P \wedge (Q \wedge R) \vdash Q$}
\RightLabel{\texttt{conjI}}
\BinaryInfC{$P \wedge (Q \wedge R) \vdash P \wedge Q$}
\AxiomC{$-$}
\RightLabel{\texttt{asm}}
\UnaryInfC{$R, P, Q \vdash R$}
\RightLabel{\texttt{rotate}}
\UnaryInfC{$Q, R, P \vdash R$}
\RightLabel{\texttt{conjE}}
\UnaryInfC{$Q \wedge R, P \vdash R$}
\RightLabel{\texttt{rotate}}
\UnaryInfC{$P, Q \wedge R \vdash R$}
\RightLabel{\texttt{conjE}}
\UnaryInfC{$P \wedge (Q \wedge R) \vdash R$}
\RightLabel{\texttt{conjI}}
\BinaryInfC{$P \wedge (Q \wedge R) \vdash (P \wedge Q) \wedge R$}
\end{prooftree}
$$

Outline

Disjunction

- Disjunction introduction

  To prove $P \vee Q$ we need to prove either P or Q:

$$\frac{\Gamma \vdash P}{\Gamma \vdash P \vee Q}\,\texttt{disjI1} \qquad \frac{\Gamma \vdash Q}{\Gamma \vdash P \vee Q}\,\texttt{disjI2}$$

- Disjunction elimination

  If one of our hypotheses is a disjunction, then we proceed by case analysis:

$$\frac{P, \Gamma \vdash R \qquad Q, \Gamma \vdash R}{P \vee Q, \Gamma \vdash R}\,\texttt{disjE}$$

Disjunction

- Disjunction introduction

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Disjunction

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Disjunction

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Disjunction

- Disjunction introduction

  To prove $P \vee Q$ we need to prove either P or Q:

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Disjunction

- **Disjunction introduction**

  To prove $P \vee Q$ we need to prove either P or Q:

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- **Disjunction elimination**

  If one of our hypotheses is a disjunction, then we proceed by case analysis:

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Disjunction

- Disjunction introduction

  To prove $P \vee Q$ we need to prove either P or Q:

$$\frac{\Gamma \vdash P}{\Gamma \vdash P \vee Q}\ \texttt{disjI1} \qquad\qquad \frac{\Gamma \vdash Q}{\Gamma \vdash P \vee Q}\ \texttt{disjI2}$$

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  If one of our hypotheses is a disjunction, then we proceed by case analysis:

$$\frac{P, \Gamma \vdash R \qquad Q, \Gamma \vdash R}{P \vee Q, \Gamma \vdash R}\ \texttt{disjE}$$

Disjunction

- Disjunction introduction

  To prove $P \vee Q$ we need to prove either P or Q:

$$\frac{\Gamma \vdash P}{\Gamma \vdash P \vee Q}\,\texttt{disjI1} \qquad \frac{\Gamma \vdash Q}{\Gamma \vdash P \vee Q}\,\texttt{disjI2}$$

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  If one of our hypotheses is a disjunction, then we proceed by case analysis:

$$\frac{P, \Gamma \vdash R \qquad Q, \Gamma \vdash R}{P \vee Q, \Gamma \vdash R}\,\texttt{disjE}$$

Example: Case Analysis

- Suppose we have an assumption that $(x = a) \vee (x = b)$ in our proof of $P(x)$.
- This suggests proceeding by case analysis:

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- This suggests proceeding by case analysis:

$$(x = a) \vee (x = b) \vdash P(x)$$

Example: Case Analysis

- Suppose we have an assumption that $(x = a) \vee (x = b)$ in our proof of $P(x)$.
- This suggests proceeding by case analysis:

$$\frac{}{(x = a) \vee (x = b) \vdash P(x)}$$

Example: Case Analysis

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Theorems as Inference Rules: the Cut

- Structural Rule: the cut.

- Example:

- We want to prove $\vdash Q \lor P$.
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- Proof

- Our previous example proved the commutativity lemma $P \vee Q \vdash Q \vee P$.
- The cut justifies using this lemma as an inference rule:
- We can now use this rule in other proofs.
- In general, if we have a proof of $P, \Gamma \vdash Q$, then we have

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Outline

Implication

- To prove an implication $P \Rightarrow Q$, we assume P and then prove Q.
- Implication introduction: $\dfrac{P, \Gamma \vdash Q}{\Gamma \vdash P \Rightarrow Q}\,\texttt{impI}$
- What if we have $P \Rightarrow Q$ as a hypothesis?
- Then we can replace it by Q. But we need to prove P in order to do this.
- Implication elimination: $\dfrac{\Gamma \vdash P \qquad Q, \Gamma \vdash R}{P \Rightarrow Q, \Gamma \vdash R}\,\texttt{impE}$
- Implication elimination was called modus ponens in antiquity.

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- The proposition

$$(P \wedge Q \Rightarrow R) \Rightarrow (P \Rightarrow (Q \Rightarrow R))$$

is a theorem of our natural deduction system.

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\AxiomC{$Q, P \vdash P \land Q$}
\AxiomC{$-$}
\RightLabel{$\mathtt{asm}$}
\UnaryInfC{$R, Q, P \vdash R$}
\RightLabel{$\mathtt{impE}$}
\BinaryInfC{$P \land Q \Rightarrow R, Q, P \vdash R$}
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$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{Q, P \vdash P} \qquad \dfrac{-}{Q, P \vdash Q}\ \texttt{asm}}{Q, P \vdash P \wedge Q}\ \texttt{conjI} \qquad \dfrac{-}{R, Q, P \vdash R}\ \texttt{asm}}{P \wedge Q \Rightarrow R, Q, P \vdash R}\ \texttt{impE}}{P, P \wedge Q \Rightarrow R, Q \vdash R}\ \texttt{rotate}}{Q, P, P \wedge Q \Rightarrow R \vdash R}\ \texttt{rotate}}{P, P \wedge Q \Rightarrow R \vdash Q \Rightarrow R}\ \texttt{impI}}{P \wedge Q \Rightarrow R \vdash P \Rightarrow (Q \Rightarrow R)}\ \texttt{impI}}{\vdash (P \wedge Q \Rightarrow R) \Rightarrow (P \Rightarrow (Q \Rightarrow R))}\ \texttt{impI}$$

Example

- The proposition

$$(P \wedge Q \Rightarrow R) \Rightarrow (P \Rightarrow (Q \Rightarrow R))$$

is a theorem of our natural deduction system.

- Proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{Q, P \vdash P}\,\texttt{rotate} \quad \dfrac{-}{Q, P \vdash Q}\,\texttt{asm}}{Q, P \vdash P \wedge Q}\,\texttt{conjI} \quad \dfrac{-}{R, Q, P \vdash R}\,\texttt{asm}}{P \wedge Q \Rightarrow R, Q, P \vdash R}\,\texttt{impE}}{P, P \wedge Q \Rightarrow R, Q \vdash R}\,\texttt{rotate}}{Q, P, P \wedge Q \Rightarrow R \vdash R}\,\texttt{rotate}}{P, P \wedge Q \Rightarrow R \vdash Q \Rightarrow R}\,\texttt{impI}}{P \wedge Q \Rightarrow R \vdash P \Rightarrow (Q \Rightarrow R)}\,\texttt{impI}}{\vdash (P \wedge Q \Rightarrow R) \Rightarrow (P \Rightarrow (Q \Rightarrow R))}\,\texttt{impI}$$

Example

- The proposition

$$(P \land Q \Rightarrow R) \Rightarrow (P \Rightarrow (Q \Rightarrow R))$$

  is a theorem of our natural deduction system.

- Proof:

$$
\begin{prooftree}
\AxiomC{$P, Q \vdash P$}
\RightLabel{$\texttt{rotate}$}
\UnaryInfC{$Q, P \vdash P$}
\AxiomC{$-$}
\RightLabel{$\texttt{asm}$}
\UnaryInfC{$Q, P \vdash Q$}
\RightLabel{$\texttt{conjI}$}
\BinaryInfC{$Q, P \vdash P \land Q$}
\AxiomC{$-$}
\RightLabel{$\texttt{asm}$}
\UnaryInfC{$R, Q, P \vdash R$}
\RightLabel{$\texttt{impE}$}
\BinaryInfC{$P \land Q \Rightarrow R, Q, P \vdash R$}
\RightLabel{$\texttt{rotate}$}
\UnaryInfC{$P, P \land Q \Rightarrow R, Q \vdash R$}
\RightLabel{$\texttt{rotate}$}
\UnaryInfC{$Q, P, P \land Q \Rightarrow R \vdash R$}
\RightLabel{$\texttt{impI}$}
\UnaryInfC{$P, P \land Q \Rightarrow R \vdash Q \Rightarrow R$}
\RightLabel{$\texttt{impI}$}
\UnaryInfC{$P \land Q \Rightarrow R \vdash P \Rightarrow (Q \Rightarrow R)$}
\RightLabel{$\texttt{impI}$}
\UnaryInfC{$\vdash (P \land Q \Rightarrow R) \Rightarrow (P \Rightarrow (Q \Rightarrow R))$}
\end{prooftree}
$$

Example

- The proposition

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  is a theorem of our natural deduction system.

- Proof:

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$P, Q \vdash P$}
\RightLabel{\texttt{rotate}}
\UnaryInfC{$Q, P \vdash P$}
\AxiomC{$-$}
\RightLabel{\texttt{asm}}
\UnaryInfC{$Q, P \vdash Q$}
\RightLabel{\texttt{conjI}}
\BinaryInfC{$Q, P \vdash P \wedge Q$}
\AxiomC{$-$}
\RightLabel{\texttt{asm}}
\UnaryInfC{$R, Q, P \vdash R$}
\RightLabel{\texttt{impE}}
\BinaryInfC{$P \wedge Q \Rightarrow R, Q, P \vdash R$}
\RightLabel{\texttt{rotate}}
\UnaryInfC{$P, P \wedge Q \Rightarrow R, Q \vdash R$}
\RightLabel{\texttt{rotate}}
\UnaryInfC{$Q, P, P \wedge Q \Rightarrow R \vdash R$}
\RightLabel{\texttt{impI}}
\UnaryInfC{$P, P \wedge Q \Rightarrow R \vdash Q \Rightarrow R$}
\RightLabel{\texttt{impI}}
\UnaryInfC{$P \wedge Q \Rightarrow R \vdash P \Rightarrow (Q \Rightarrow R)$}
\RightLabel{\texttt{impI}}
\UnaryInfC{$\vdash (P \wedge Q \Rightarrow R) \Rightarrow (P \Rightarrow (Q \Rightarrow R))$}
\end{prooftree}
$$

Example

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is a theorem of our natural deduction system.

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$$
\begin{prooftree}
\AxiomC{}
\RightLabel{\texttt{asm}}
\UnaryInfC{$P, Q \vdash P$}
\RightLabel{\texttt{rotate}}
\UnaryInfC{$Q, P \vdash P$}
\AxiomC{$-$}
\RightLabel{\texttt{asm}}
\UnaryInfC{$Q, P \vdash Q$}
\RightLabel{\texttt{conjI}}
\BinaryInfC{$Q, P \vdash P \wedge Q$}
\AxiomC{$-$}
\RightLabel{\texttt{asm}}
\UnaryInfC{$R, Q, P \vdash R$}
\RightLabel{\texttt{impE}}
\BinaryInfC{$P \wedge Q \Rightarrow R, Q, P \vdash R$}
\RightLabel{\texttt{rotate}}
\UnaryInfC{$P, P \wedge Q \Rightarrow R, Q \vdash R$}
\RightLabel{\texttt{rotate}}
\UnaryInfC{$Q, P, P \wedge Q \Rightarrow R \vdash R$}
\RightLabel{\texttt{impI}}
\UnaryInfC{$P, P \wedge Q \Rightarrow R \vdash Q \Rightarrow R$}
\RightLabel{\texttt{impI}}
\UnaryInfC{$P \wedge Q \Rightarrow R \vdash P \Rightarrow (Q \Rightarrow R)$}
\RightLabel{\texttt{impI}}
\UnaryInfC{$\vdash (P \wedge Q \Rightarrow R) \Rightarrow (P \Rightarrow (Q \Rightarrow R))$}
\end{prooftree}
$$

Example

- The proposition

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is a theorem of our natural deduction system.

- Proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{-}{P, Q \vdash P}\,\texttt{asm}}{Q, P \vdash P}\,\texttt{rotate} \qquad \dfrac{-}{Q, P \vdash Q}\,\texttt{asm}}{Q, P \vdash P \wedge Q}\,\texttt{conjI} \qquad \dfrac{-}{R, Q, P \vdash R}\,\texttt{asm}}{P \wedge Q \Rightarrow R, Q, P \vdash R}\,\texttt{impE}}{P, P \wedge Q \Rightarrow R, Q \vdash R}\,\texttt{rotate}}{Q, P, P \wedge Q \Rightarrow R \vdash R}\,\texttt{rotate}}{P, P \wedge Q \Rightarrow R \vdash Q \Rightarrow R}\,\texttt{impI}}{P \wedge Q \Rightarrow R \vdash P \Rightarrow (Q \Rightarrow R)}\,\texttt{impI}}{\vdash (P \wedge Q \Rightarrow R) \Rightarrow (P \Rightarrow (Q \Rightarrow R))}\,\texttt{impI}$$

Example: Implication is Transitive

- We want to prove: $(P \Rightarrow Q) \land (Q \Rightarrow R) \Rightarrow (P \Rightarrow R)$.
- Proof:

Example: Implication is Transitive

- We want to prove: $(P \Rightarrow Q) \wedge (Q \Rightarrow R) \Rightarrow (P \Rightarrow R)$.
- Proof:

Example: Implication is Transitive

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Example: Implication is Transitive

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- Proof:

$$\vdash (P \Rightarrow Q) \land (Q \Rightarrow R) \Rightarrow (P \Rightarrow R)$$

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- We want to prove: $(P \Rightarrow Q) \land (Q \Rightarrow R) \Rightarrow (P \Rightarrow R)$.
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$$\frac{}{\vdash (P \Rightarrow Q) \land (Q \Rightarrow R) \Rightarrow (P \Rightarrow R)}$$

Example: Implication is Transitive

- We want to prove: $(P \Rightarrow Q) \wedge (Q \Rightarrow R) \Rightarrow (P \Rightarrow R)$.
- Proof:

$$\dfrac{}{\vdash (P \Rightarrow Q) \wedge (Q \Rightarrow R) \Rightarrow (P \Rightarrow R)}\ \texttt{impI}$$

Example: Implication is Transitive

- We want to prove: $(P \Rightarrow Q) \wedge (Q \Rightarrow R) \Rightarrow (P \Rightarrow R)$.
- Proof:

$$\frac{(P \Rightarrow Q) \wedge (Q \Rightarrow R) \vdash P \Rightarrow R}{\vdash (P \Rightarrow Q) \wedge (Q \Rightarrow R) \Rightarrow (P \Rightarrow R)}\ \texttt{impI}$$

Example: Implication is Transitive

- We want to prove: $(P \Rightarrow Q) \wedge (Q \Rightarrow R) \Rightarrow (P \Rightarrow R)$.
- Proof:

$$\dfrac{\dfrac{}{(P \Rightarrow Q) \wedge (Q \Rightarrow R) \vdash P \Rightarrow R}}{\vdash (P \Rightarrow Q) \wedge (Q \Rightarrow R) \Rightarrow (P \Rightarrow R)}\ \mathtt{impI}$$

Example: Implication is Transitive

- We want to prove: $(P \Rightarrow Q) \land (Q \Rightarrow R) \Rightarrow (P \Rightarrow R)$.
- Proof:

$$\dfrac{\dfrac{}{(P \Rightarrow Q) \land (Q \Rightarrow R) \vdash P \Rightarrow R}\;\texttt{impI}}{\vdash (P \Rightarrow Q) \land (Q \Rightarrow R) \Rightarrow (P \Rightarrow R)}\;\texttt{impI}$$

Example: Implication is Transitive

- We want to prove: $(P \Rightarrow Q) \wedge (Q \Rightarrow R) \Rightarrow (P \Rightarrow R)$.
- Proof:

$$\dfrac{\dfrac{P, (P \Rightarrow Q) \wedge (Q \Rightarrow R) \vdash R}{(P \Rightarrow Q) \wedge (Q \Rightarrow R) \vdash P \Rightarrow R}\ \texttt{impI}}{\vdash (P \Rightarrow Q) \wedge (Q \Rightarrow R) \Rightarrow (P \Rightarrow R)}\ \texttt{impI}$$

Example: Implication is Transitive

- We want to prove: $(P \Rightarrow Q) \wedge (Q \Rightarrow R) \Rightarrow (P \Rightarrow R)$.
- Proof:

$$\cfrac{\cfrac{\overline{P, (P \Rightarrow Q) \wedge (Q \Rightarrow R) \vdash R}}{(P \Rightarrow Q) \wedge (Q \Rightarrow R) \vdash P \Rightarrow R}\ \texttt{impI}}{\vdash (P \Rightarrow Q) \wedge (Q \Rightarrow R) \Rightarrow (P \Rightarrow R)}\ \texttt{impI}$$

Example: Implication is Transitive

- We want to prove: $(P \Rightarrow Q) \wedge (Q \Rightarrow R) \Rightarrow (P \Rightarrow R)$.
- Proof:

$$\dfrac{\dfrac{\overline{P, (P \Rightarrow Q) \wedge (Q \Rightarrow R) \vdash R}\ \texttt{rotate}}{(P \Rightarrow Q) \wedge (Q \Rightarrow R) \vdash P \Rightarrow R}\ \texttt{impI}}{\vdash (P \Rightarrow Q) \wedge (Q \Rightarrow R) \Rightarrow (P \Rightarrow R)}\ \texttt{impI}$$

Example: Implication is Transitive

- We want to prove: $(P \Rightarrow Q) \wedge (Q \Rightarrow R) \Rightarrow (P \Rightarrow R)$.
- Proof:

$$\dfrac{\dfrac{\dfrac{(P \Rightarrow Q) \wedge (Q \Rightarrow R), P \vdash R}{P, (P \Rightarrow Q) \wedge (Q \Rightarrow R) \vdash R}\ \texttt{rotate}}{(P \Rightarrow Q) \wedge (Q \Rightarrow R) \vdash P \Rightarrow R}\ \texttt{impI}}{\vdash (P \Rightarrow Q) \wedge (Q \Rightarrow R) \Rightarrow (P \Rightarrow R)}\ \texttt{impI}$$

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$$\dfrac{\dfrac{\dfrac{\overline{(P \Rightarrow Q) \wedge (Q \Rightarrow R), P \vdash R}}{P, (P \Rightarrow Q) \wedge (Q \Rightarrow R) \vdash R}\ \texttt{rotate}}{(P \Rightarrow Q) \wedge (Q \Rightarrow R) \vdash P \Rightarrow R}\ \texttt{impI}}{\vdash (P \Rightarrow Q) \wedge (Q \Rightarrow R) \Rightarrow (P \Rightarrow R)}\ \texttt{impI}$$

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$$
\dfrac{\dfrac{\dfrac{\dfrac{}{(P \Rightarrow Q) \wedge (Q \Rightarrow R), P \vdash R}\ \texttt{conjE}}{P, (P \Rightarrow Q) \wedge (Q \Rightarrow R) \vdash R}\ \texttt{rotate}}{(P \Rightarrow Q) \wedge (Q \Rightarrow R) \vdash P \Rightarrow R}\ \texttt{impI}}{\vdash (P \Rightarrow Q) \wedge (Q \Rightarrow R) \Rightarrow (P \Rightarrow R)}\ \texttt{impI}
$$

Example: Implication is Transitive

- We want to prove: $(P \Rightarrow Q) \land (Q \Rightarrow R) \Rightarrow (P \Rightarrow R)$.
- Proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{P \Rightarrow Q, Q \Rightarrow R, P \vdash R}{(P \Rightarrow Q) \land (Q \Rightarrow R), P \vdash R}\ \texttt{conjE}}{P, (P \Rightarrow Q) \land (Q \Rightarrow R) \vdash R}\ \texttt{rotate}}{(P \Rightarrow Q) \land (Q \Rightarrow R) \vdash P \Rightarrow R}\ \texttt{impI}}{\vdash (P \Rightarrow Q) \land (Q \Rightarrow R) \Rightarrow (P \Rightarrow R)}\ \texttt{impI}$$

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\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P \Rightarrow Q, Q \Rightarrow R, P \vdash R}}{(P \Rightarrow Q) \wedge (Q \Rightarrow R), P \vdash R}\ \texttt{conjE}}{P, (P \Rightarrow Q) \wedge (Q \Rightarrow R) \vdash R}\ \texttt{rotate}}{(P \Rightarrow Q) \wedge (Q \Rightarrow R) \vdash P \Rightarrow R}\ \texttt{impI}}{\vdash (P \Rightarrow Q) \wedge (Q \Rightarrow R) \Rightarrow (P \Rightarrow R)}\ \texttt{impI}
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$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P \Rightarrow Q, Q \Rightarrow R, P \vdash R}\;\texttt{impE}}{(P \Rightarrow Q) \wedge (Q \Rightarrow R), P \vdash R}\;\texttt{conjE}}{P, (P \Rightarrow Q) \wedge (Q \Rightarrow R) \vdash R}\;\texttt{rotate}}{(P \Rightarrow Q) \wedge (Q \Rightarrow R) \vdash P \Rightarrow R}\;\texttt{impI}}{\vdash (P \Rightarrow Q) \wedge (Q \Rightarrow R) \Rightarrow (P \Rightarrow R)}\;\texttt{impI}
$$

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$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{Q \Rightarrow R, P \vdash P \qquad Q, Q \Rightarrow R, P \vdash R}{P \Rightarrow Q, Q \Rightarrow R, P \vdash R}\ \texttt{impE}}{(P \Rightarrow Q) \wedge (Q \Rightarrow R), P \vdash R}\ \texttt{conjE}}{P, (P \Rightarrow Q) \wedge (Q \Rightarrow R) \vdash R}\ \texttt{rotate}}{(P \Rightarrow Q) \wedge (Q \Rightarrow R) \vdash P \Rightarrow R}\ \texttt{impI}}{\vdash (P \Rightarrow Q) \wedge (Q \Rightarrow R) \Rightarrow (P \Rightarrow R)}\ \texttt{impI}
$$

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$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{Q \Rightarrow R, P \vdash P \qquad \dfrac{}{Q, Q \Rightarrow R, P \vdash R}}{P \Rightarrow Q, Q \Rightarrow R, P \vdash R}\,\texttt{impE}}{(P \Rightarrow Q) \wedge (Q \Rightarrow R), P \vdash R}\,\texttt{conjE}}{P, (P \Rightarrow Q) \wedge (Q \Rightarrow R) \vdash R}\,\texttt{rotate}}{(P \Rightarrow Q) \wedge (Q \Rightarrow R) \vdash P \Rightarrow R}\,\texttt{impI}}{\vdash (P \Rightarrow Q) \wedge (Q \Rightarrow R) \Rightarrow (P \Rightarrow R)}\,\texttt{impI}$$

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\dfrac{\dfrac{\dfrac{\dfrac{Q \Rightarrow R, P \vdash P \qquad \dfrac{}{Q, Q \Rightarrow R, P \vdash R}\,\texttt{rotate}}{P \Rightarrow Q, Q \Rightarrow R, P \vdash R}\,\texttt{impE}}{(P \Rightarrow Q) \wedge (Q \Rightarrow R), P \vdash R}\,\texttt{conjE}}{\dfrac{P, (P \Rightarrow Q) \wedge (Q \Rightarrow R) \vdash R}{(P \Rightarrow Q) \wedge (Q \Rightarrow R) \vdash P \Rightarrow R}\,\texttt{impI}}\,\texttt{rotate}}{\vdash (P \Rightarrow Q) \wedge (Q \Rightarrow R) \Rightarrow (P \Rightarrow R)}\,\texttt{impI}
$$

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$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{Q \Rightarrow R, P \vdash P \qquad \dfrac{Q \Rightarrow R, P, Q \vdash R}{Q, Q \Rightarrow R, P \vdash R}\,\texttt{rotate}}{P \Rightarrow Q, Q \Rightarrow R, P \vdash R}\,\texttt{impE}}{(P \Rightarrow Q) \wedge (Q \Rightarrow R), P \vdash R}\,\texttt{conjE}}{P, (P \Rightarrow Q) \wedge (Q \Rightarrow R) \vdash R}\,\texttt{rotate}}{(P \Rightarrow Q) \wedge (Q \Rightarrow R) \vdash P \Rightarrow R}\,\texttt{impI}}{\vdash (P \Rightarrow Q) \wedge (Q \Rightarrow R) \Rightarrow (P \Rightarrow R)}\,\texttt{impI}
$$

Example: Implication is Transitive

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$$
\dfrac{\dfrac{\dfrac{\dfrac{Q \Rightarrow R, P \vdash P \qquad \dfrac{\dfrac{}{Q \Rightarrow R, P, Q \vdash R}}{Q, Q \Rightarrow R, P \vdash R}\;\texttt{rotate}}{P \Rightarrow Q, Q \Rightarrow R, P \vdash R}\;\texttt{impE}}{(P \Rightarrow Q) \land (Q \Rightarrow R), P \vdash R}\;\texttt{conjE}}{P, (P \Rightarrow Q) \land (Q \Rightarrow R) \vdash R}\;\texttt{rotate}
$$

$$
\dfrac{\dfrac{P, (P \Rightarrow Q) \land (Q \Rightarrow R) \vdash R}{(P \Rightarrow Q) \land (Q \Rightarrow R) \vdash P \Rightarrow R}\;\texttt{impI}}{\vdash (P \Rightarrow Q) \land (Q \Rightarrow R) \Rightarrow (P \Rightarrow R)}\;\texttt{impI}
$$

Example: Implication is Transitive

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$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{Q \Rightarrow R, P \vdash P \qquad \dfrac{\dfrac{}{Q \Rightarrow R, P, Q \vdash R}\;\texttt{impE}}{Q, Q \Rightarrow R, P \vdash R}\;\texttt{rotate}}{P \Rightarrow Q, Q \Rightarrow R, P \vdash R}\;\texttt{impE}}{(P \Rightarrow Q) \wedge (Q \Rightarrow R), P \vdash R}\;\texttt{conjE}}{P, (P \Rightarrow Q) \wedge (Q \Rightarrow R) \vdash R}\;\texttt{rotate}}{(P \Rightarrow Q) \wedge (Q \Rightarrow R) \vdash P \Rightarrow R}\;\texttt{impI}}{\vdash (P \Rightarrow Q) \wedge (Q \Rightarrow R) \Rightarrow (P \Rightarrow R)}\;\texttt{impI}
$$

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\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{Q \Rightarrow R, P \vdash P \qquad \dfrac{\dfrac{P, Q \vdash Q \qquad R, P, Q \vdash R}{Q \Rightarrow R, P, Q \vdash R}\ \texttt{impE}}{Q, Q \Rightarrow R, P \vdash R}\ \texttt{rotate}}{P \Rightarrow Q, Q \Rightarrow R, P \vdash R}\ \texttt{impE}}{(P \Rightarrow Q) \wedge (Q \Rightarrow R), P \vdash R}\ \texttt{conjE}}{P, (P \Rightarrow Q) \wedge (Q \Rightarrow R) \vdash R}\ \texttt{rotate}}{(P \Rightarrow Q) \wedge (Q \Rightarrow R) \vdash P \Rightarrow R}\ \texttt{impI}}{\vdash (P \Rightarrow Q) \wedge (Q \Rightarrow R) \Rightarrow (P \Rightarrow R)}\ \texttt{impI}
$$

Example: Implication is Transitive

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$$
\dfrac{
  \dfrac{
    \dfrac{
      Q \Rightarrow R, P \vdash P
      \qquad
      \dfrac{
        \dfrac{
          P, Q \vdash Q
          \qquad
          \dfrac{}{R, P, Q \vdash R}
        }{Q \Rightarrow R, P, Q \vdash R}\;\mathtt{impE}
      }{Q, Q \Rightarrow R, P \vdash R}\;\mathtt{rotate}
    }{P \Rightarrow Q, Q \Rightarrow R, P \vdash R}\;\mathtt{impE}
  }{
    \dfrac{
      \dfrac{(P \Rightarrow Q) \wedge (Q \Rightarrow R), P \vdash R}{P, (P \Rightarrow Q) \wedge (Q \Rightarrow R) \vdash R}\;\mathtt{rotate}
    }{(P \Rightarrow Q) \wedge (Q \Rightarrow R) \vdash P \Rightarrow R}\;\mathtt{impI}
  }\;\mathtt{conjE}
}{\vdash (P \Rightarrow Q) \wedge (Q \Rightarrow R) \Rightarrow (P \Rightarrow R)}\;\mathtt{impI}
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\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{Q \Rightarrow R, P \vdash P \qquad \dfrac{\dfrac{P, Q \vdash Q \qquad \dfrac{}{R, P, Q \vdash R}\;\texttt{asm}}{Q \Rightarrow R, P, Q \vdash R}\;\texttt{impE}}{Q, Q \Rightarrow R, P \vdash R}\;\texttt{rotate}}{P \Rightarrow Q, Q \Rightarrow R, P \vdash R}\;\texttt{impE}}{(P \Rightarrow Q) \wedge (Q \Rightarrow R), P \vdash R}\;\texttt{conjE}}{P, (P \Rightarrow Q) \wedge (Q \Rightarrow R) \vdash R}\;\texttt{rotate}}{(P \Rightarrow Q) \wedge (Q \Rightarrow R) \vdash P \Rightarrow R}\;\texttt{impI}}{\vdash (P \Rightarrow Q) \wedge (Q \Rightarrow R) \Rightarrow (P \Rightarrow R)}\;\texttt{impI}
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Example: Implication is Transitive

- We want to prove: $(P \Rightarrow Q) \land (Q \Rightarrow R) \Rightarrow (P \Rightarrow R)$.
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$$
\begin{prooftree}
\AxiomC{$Q \Rightarrow R, P \vdash P$}
\AxiomC{$P, Q \vdash Q$}
\AxiomC{}
\RightLabel{\texttt{asm}}
\UnaryInfC{$R, P, Q \vdash R$}
\RightLabel{\texttt{impE}}
\BinaryInfC{$Q \Rightarrow R, P, Q \vdash R$}
\RightLabel{\texttt{rotate}}
\UnaryInfC{$Q, Q \Rightarrow R, P \vdash R$}
\RightLabel{\texttt{impE}}
\BinaryInfC{$P \Rightarrow Q, Q \Rightarrow R, P \vdash R$}
\RightLabel{\texttt{conjE}}
\UnaryInfC{$(P \Rightarrow Q) \land (Q \Rightarrow R), P \vdash R$}
\RightLabel{\texttt{rotate}}
\UnaryInfC{$P, (P \Rightarrow Q) \land (Q \Rightarrow R) \vdash R$}
\RightLabel{\texttt{impI}}
\UnaryInfC{$(P \Rightarrow Q) \land (Q \Rightarrow R) \vdash P \Rightarrow R$}
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\end{prooftree}
$$

Example: Implication is Transitive

- We want to prove: $(P \Rightarrow Q) \wedge (Q \Rightarrow R) \Rightarrow (P \Rightarrow R)$.
- Proof:

$$
\begin{prooftree}
\AxiomC{$Q \Rightarrow R, P \vdash P$}
\AxiomC{}
\UnaryInfC{$P, Q \vdash Q$}
\AxiomC{}
\RightLabel{$\texttt{asm}$}
\UnaryInfC{$R, P, Q \vdash R$}
\RightLabel{$\texttt{impE}$}
\BinaryInfC{$Q \Rightarrow R, P, Q \vdash R$}
\RightLabel{$\texttt{rotate}$}
\UnaryInfC{$Q, Q \Rightarrow R, P \vdash R$}
\RightLabel{$\texttt{impE}$}
\BinaryInfC{$P \Rightarrow Q, Q \Rightarrow R, P \vdash R$}
\RightLabel{$\texttt{conjE}$}
\UnaryInfC{$(P \Rightarrow Q) \wedge (Q \Rightarrow R), P \vdash R$}
\RightLabel{$\texttt{rotate}$}
\UnaryInfC{$P, (P \Rightarrow Q) \wedge (Q \Rightarrow R) \vdash R$}
\RightLabel{$\texttt{impI}$}
\UnaryInfC{$(P \Rightarrow Q) \wedge (Q \Rightarrow R) \vdash P \Rightarrow R$}
\RightLabel{$\texttt{impI}$}
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$$

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\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{Q \Rightarrow R, P \vdash P \qquad \dfrac{\dfrac{\dfrac{}{P, Q \vdash Q}\,\texttt{rotate} \qquad \dfrac{-}{R, P, Q \vdash R}\,\texttt{asm}}{Q \Rightarrow R, P, Q \vdash R}\,\texttt{impE}}{Q, Q \Rightarrow R, P \vdash R}\,\texttt{rotate}}{P \Rightarrow Q, Q \Rightarrow R, P \vdash R}\,\texttt{impE}}{(P \Rightarrow Q) \wedge (Q \Rightarrow R), P \vdash R}\,\texttt{conjE}}{P, (P \Rightarrow Q) \wedge (Q \Rightarrow R) \vdash R}\,\texttt{rotate}}{(P \Rightarrow Q) \wedge (Q \Rightarrow R) \vdash P \Rightarrow R}\,\texttt{impI}}{\vdash (P \Rightarrow Q) \wedge (Q \Rightarrow R) \Rightarrow (P \Rightarrow R)}\,\texttt{impI}
$$

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\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{Q \Rightarrow R, P \vdash P \qquad \cfrac{\cfrac{\cfrac{Q, P \vdash Q}{P, Q \vdash Q}\,\texttt{rotate} \qquad \cfrac{-}{R, P, Q \vdash R}\,\texttt{asm}}{Q \Rightarrow R, P, Q \vdash R}\,\texttt{impE}}{Q, Q \Rightarrow R, P \vdash R}\,\texttt{rotate}}{P \Rightarrow Q, Q \Rightarrow R, P \vdash R}\,\texttt{impE}}{(P \Rightarrow Q) \wedge (Q \Rightarrow R), P \vdash R}\,\texttt{conjE}}{P, (P \Rightarrow Q) \wedge (Q \Rightarrow R) \vdash R}\,\texttt{rotate}}{(P \Rightarrow Q) \wedge (Q \Rightarrow R) \vdash P \Rightarrow R}\,\texttt{impI}}{\vdash (P \Rightarrow Q) \wedge (Q \Rightarrow R) \Rightarrow (P \Rightarrow R)}\,\texttt{impI}
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- Proof:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{Q \Rightarrow R, P \vdash P \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{Q, P \vdash Q}}{P, Q \vdash Q}\;\mathtt{rotate} \qquad \dfrac{-}{R, P, Q \vdash R}\;\mathtt{asm}}{Q \Rightarrow R, P, Q \vdash R}\;\mathtt{impE}}{Q, Q \Rightarrow R, P \vdash R}\;\mathtt{rotate}}{P \Rightarrow Q, Q \Rightarrow R, P \vdash R}\;\mathtt{impE}}{(P \Rightarrow Q) \wedge (Q \Rightarrow R), P \vdash R}\;\mathtt{conjE}}{P, (P \Rightarrow Q) \wedge (Q \Rightarrow R) \vdash R}\;\mathtt{rotate}}{(P \Rightarrow Q) \wedge (Q \Rightarrow R) \vdash P \Rightarrow R}\;\mathtt{impI}}{\vdash (P \Rightarrow Q) \wedge (Q \Rightarrow R) \Rightarrow (P \Rightarrow R)}\;\mathtt{impI}
$$

Example: Implication is Transitive

- We want to prove: $(P \Rightarrow Q) \wedge (Q \Rightarrow R) \Rightarrow (P \Rightarrow R)$.
- Proof:

$$
\begin{prooftree}
\AxiomC{$Q \Rightarrow R, P \vdash P$}
\AxiomC{}
\RightLabel{$\texttt{asm}$}
\UnaryInfC{$Q, P \vdash Q$}
\RightLabel{$\texttt{rotate}$}
\UnaryInfC{$P, Q \vdash Q$}
\AxiomC{$-$}
\RightLabel{$\texttt{asm}$}
\UnaryInfC{$R, P, Q \vdash R$}
\RightLabel{$\texttt{impE}$}
\BinaryInfC{$Q \Rightarrow R, P, Q \vdash R$}
\RightLabel{$\texttt{rotate}$}
\UnaryInfC{$Q, Q \Rightarrow R, P \vdash R$}
\RightLabel{$\texttt{impE}$}
\BinaryInfC{$P \Rightarrow Q, Q \Rightarrow R, P \vdash R$}
\RightLabel{$\texttt{conjE}$}
\UnaryInfC{$(P \Rightarrow Q) \wedge (Q \Rightarrow R), P \vdash R$}
\RightLabel{$\texttt{rotate}$}
\UnaryInfC{$P, (P \Rightarrow Q) \wedge (Q \Rightarrow R) \vdash R$}
\RightLabel{$\texttt{impI}$}
\UnaryInfC{$(P \Rightarrow Q) \wedge (Q \Rightarrow R) \vdash P \Rightarrow R$}
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\begin{prooftree}
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\AxiomC{$-$}
\RightLabel{$\texttt{asm}$}
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\RightLabel{$\texttt{rotate}$}
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\UnaryInfC{$R, P, Q \vdash R$}
\RightLabel{$\texttt{impE}$}
\BinaryInfC{$Q \Rightarrow R, P, Q \vdash R$}
\RightLabel{$\texttt{rotate}$}
\UnaryInfC{$Q, Q \Rightarrow R, P \vdash R$}
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\RightLabel{$\texttt{rotate}$}
\UnaryInfC{$P, (P \Rightarrow Q) \wedge (Q \Rightarrow R) \vdash R$}
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\dfrac{
  \dfrac{
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      \dfrac{
        \dfrac{
          \dfrac{}{Q \Rightarrow R, P \vdash P}\,\texttt{rotate}
          \qquad
          \dfrac{
            \dfrac{
              \dfrac{
                \dfrac{\dfrac{-}{Q, P \vdash Q}\,\texttt{asm}}{P, Q \vdash Q}\,\texttt{rotate}
                \qquad
                \dfrac{-}{R, P, Q \vdash R}\,\texttt{asm}
              }{Q \Rightarrow R, P, Q \vdash R}\,\texttt{impE}
            }{Q, Q \Rightarrow R, P \vdash R}\,\texttt{rotate}
          }{}
        }{P \Rightarrow Q, Q \Rightarrow R, P \vdash R}\,\texttt{impE}
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Outline

Equivalence

- The equivalence $P \Leftrightarrow Q$ is the bi-implication $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$.
- This explains the two proof rules, which are related to those for conjunction:

$$\frac{\Gamma \vdash P \Rightarrow Q \qquad \Gamma \vdash Q \Rightarrow P}{\Gamma \vdash P \Leftrightarrow Q}\,\mathtt{iffI} \qquad \frac{P \Rightarrow Q, Q \Rightarrow P, \Gamma \vdash R}{P \Leftrightarrow Q, \Gamma \vdash R}\,\mathtt{iffE}$$

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Example: Rule of Subsumption

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$$P \Rightarrow Q \vdash P \land Q \Leftrightarrow P$$

Example: Rule of Subsumption

$$\frac{}{P \Rightarrow Q \vdash P \wedge Q \Leftrightarrow P}$$

Example: Rule of Subsumption

$$\frac{}{P \Rightarrow Q \vdash P \wedge Q \Leftrightarrow P}\ \texttt{iffI}$$

Example: Rule of Subsumption

$$\dfrac{P \Rightarrow Q \vdash P \wedge Q \Rightarrow P \qquad P \Rightarrow Q \vdash P \Rightarrow P \wedge Q}{P \Rightarrow Q \vdash P \wedge Q \Leftrightarrow P}\ \texttt{iffI}$$

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$$\dfrac{P \Rightarrow Q \vdash P \wedge Q \Rightarrow P \qquad \dfrac{}{P \Rightarrow Q \vdash P \Rightarrow P \wedge Q}}{P \Rightarrow Q \vdash P \wedge Q \Leftrightarrow P}\ \texttt{iffI}$$

Example: Rule of Subsumption

$$\dfrac{P \Rightarrow Q \vdash P \wedge Q \Rightarrow P \qquad \dfrac{}{P \Rightarrow Q \vdash P \Rightarrow P \wedge Q}\,\texttt{impI}}{P \Rightarrow Q \vdash P \wedge Q \Leftrightarrow P}\,\texttt{iffI}$$

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$$\dfrac{P \Rightarrow Q \vdash P \wedge Q \Rightarrow P \qquad \dfrac{P, P \Rightarrow Q \vdash P \wedge Q}{P \Rightarrow Q \vdash P \Rightarrow P \wedge Q}\;\texttt{impI}}{P \Rightarrow Q \vdash P \wedge Q \Leftrightarrow P}\;\texttt{iffI}$$

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$$\dfrac{P \Rightarrow Q \vdash P \wedge Q \Rightarrow P \qquad \dfrac{\dfrac{}{P, P \Rightarrow Q \vdash P \wedge Q}\,\texttt{rotate}}{P \Rightarrow Q \vdash P \Rightarrow P \wedge Q}\,\texttt{impI}}{P \Rightarrow Q \vdash P \wedge Q \Leftrightarrow P}\,\texttt{iffI}$$

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$$
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$$

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$$
\dfrac{P \Rightarrow Q \vdash P \land Q \Rightarrow P \qquad \dfrac{\dfrac{\dfrac{P \vdash P \qquad Q, P \vdash P \land Q}{P \Rightarrow Q, P \vdash P \land Q}\;\texttt{impE}}{P, P \Rightarrow Q \vdash P \land Q}\;\texttt{rotate}}{P \Rightarrow Q \vdash P \Rightarrow P \land Q}\;\texttt{impI}}{P \Rightarrow Q \vdash P \land Q \Leftrightarrow P}\;\texttt{iffI}
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$$
\dfrac{P \Rightarrow Q \vdash P \wedge Q \Rightarrow P \qquad \dfrac{\dfrac{\dfrac{P \vdash P \qquad \dfrac{}{Q, P \vdash P \wedge Q}\,\texttt{conjI}}{P \Rightarrow Q, P \vdash P \wedge Q}\,\texttt{impE}}{P, P \Rightarrow Q \vdash P \wedge Q}\,\texttt{rotate}}{P \Rightarrow Q \vdash P \Rightarrow P \wedge Q}\,\texttt{impI}}{P \Rightarrow Q \vdash P \wedge Q \Leftrightarrow P}\,\texttt{iffI}
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$$
\begin{prooftree}
\AxiomC{$P \Rightarrow Q \vdash P \wedge Q \Rightarrow P$}
\AxiomC{$P \vdash P$}
\AxiomC{$Q, P \vdash P$}
\AxiomC{}
\UnaryInfC{$Q, P \vdash Q$}
\RightLabel{$\texttt{conjI}$}
\BinaryInfC{$Q, P \vdash P \wedge Q$}
\RightLabel{$\texttt{impE}$}
\BinaryInfC{$P \Rightarrow Q, P \vdash P \wedge Q$}
\RightLabel{$\texttt{rotate}$}
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$$

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\dfrac{P \Rightarrow Q \vdash P \wedge Q \Rightarrow P \qquad \dfrac{\dfrac{\dfrac{P \vdash P \qquad \dfrac{Q, P \vdash P \qquad \dfrac{}{Q, P \vdash Q}\,\texttt{asm}}{Q, P \vdash P \wedge Q}\,\texttt{conjI}}{P \Rightarrow Q, P \vdash P \wedge Q}\,\texttt{impE}}{P, P \Rightarrow Q \vdash P \wedge Q}\,\texttt{rotate}}{P \Rightarrow Q \vdash P \Rightarrow P \wedge Q}\,\texttt{impI}}{P \Rightarrow Q \vdash P \wedge Q \Leftrightarrow P}\,\texttt{iffI}
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\begin{prooftree}
\AxiomC{$P \Rightarrow Q \vdash P \wedge Q \Rightarrow P$}
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\RightLabel{asm}
\UnaryInfC{$P \vdash P$}
\AxiomC{$-$}
\RightLabel{asm}
\UnaryInfC{$P, Q \vdash P$}
\RightLabel{rotate}
\UnaryInfC{$Q, P \vdash P$}
\AxiomC{$-$}
\RightLabel{asm}
\UnaryInfC{$Q, P \vdash Q$}
\RightLabel{conjI}
\BinaryInfC{$Q, P \vdash P \wedge Q$}
\RightLabel{impE}
\BinaryInfC{$P \Rightarrow Q, P \vdash P \wedge Q$}
\RightLabel{rotate}
\UnaryInfC{$P, P \Rightarrow Q \vdash P \wedge Q$}
\RightLabel{impI}
\UnaryInfC{$P \Rightarrow Q \vdash P \Rightarrow P \wedge Q$}
\RightLabel{iffI}
\BinaryInfC{$P \Rightarrow Q \vdash P \wedge Q \Leftrightarrow P$}
\end{prooftree}
$$

Example: Rule of Subsumption

$$\dfrac{\dfrac{}{P \Rightarrow Q \vdash P \land Q \Rightarrow P}\,\texttt{impI} \qquad \dfrac{\dfrac{\dfrac{\dfrac{-}{P \vdash P}\,\texttt{asm} \quad \dfrac{\dfrac{\dfrac{-}{P, Q \vdash P}\,\texttt{asm}}{Q, P \vdash P}\,\texttt{rotate} \quad \dfrac{-}{Q, P \vdash Q}\,\texttt{asm}}{Q, P \vdash P \land Q}\,\texttt{conjI}}{P \Rightarrow Q, P \vdash P \land Q}\,\texttt{impE}}{P, P \Rightarrow Q \vdash P \land Q}\,\texttt{rotate}}{P \Rightarrow Q \vdash P \Rightarrow P \land Q}\,\texttt{impI}}{P \Rightarrow Q \vdash P \land Q \Leftrightarrow P}\,\texttt{iffI}$$

Example: Rule of Subsumption

$$
\dfrac{
  \dfrac{P \wedge Q, P \Rightarrow Q \vdash P}{P \Rightarrow Q \vdash P \wedge Q \Rightarrow P}\;\texttt{impI}
  \qquad
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{-}{P \vdash P}\;\texttt{asm}
        \quad
        \dfrac{
          \dfrac{\dfrac{-}{P, Q \vdash P}\;\texttt{asm}}{Q, P \vdash P}\;\texttt{rotate}
          \quad
          \dfrac{-}{Q, P \vdash Q}\;\texttt{asm}
        }{Q, P \vdash P \wedge Q}\;\texttt{conjI}
      }{P \Rightarrow Q, P \vdash P \wedge Q}\;\texttt{impE}
    }{P, P \Rightarrow Q \vdash P \wedge Q}\;\texttt{rotate}
  }{P \Rightarrow Q \vdash P \Rightarrow P \wedge Q}\;\texttt{impI}
}{P \Rightarrow Q \vdash P \wedge Q \Leftrightarrow P}\;\texttt{iffI}
$$

Example: Rule of Subsumption

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$P \wedge Q, P \Rightarrow Q \vdash P$}
\RightLabel{\texttt{impI}}
\UnaryInfC{$P \Rightarrow Q \vdash P \wedge Q \Rightarrow P$}
\AxiomC{$-$}
\RightLabel{\texttt{asm}}
\UnaryInfC{$P \vdash P$}
\AxiomC{$-$}
\RightLabel{\texttt{asm}}
\UnaryInfC{$P, Q \vdash P$}
\RightLabel{\texttt{rotate}}
\UnaryInfC{$Q, P \vdash P$}
\AxiomC{$-$}
\RightLabel{\texttt{asm}}
\UnaryInfC{$Q, P \vdash Q$}
\RightLabel{\texttt{conjI}}
\BinaryInfC{$Q, P \vdash P \wedge Q$}
\RightLabel{\texttt{impE}}
\BinaryInfC{$P \Rightarrow Q, P \vdash P \wedge Q$}
\RightLabel{\texttt{rotate}}
\UnaryInfC{$P, P \Rightarrow Q \vdash P \wedge Q$}
\RightLabel{\texttt{impI}}
\UnaryInfC{$P \Rightarrow Q \vdash P \Rightarrow P \wedge Q$}
\RightLabel{\texttt{iffI}}
\BinaryInfC{$P \Rightarrow Q \vdash P \wedge Q \Leftrightarrow P$}
\end{prooftree}
$$

Example: Rule of Subsumption

$$
\dfrac{
\dfrac{
\dfrac{}{P \land Q, P \Rightarrow Q \vdash P}\;\mathtt{conjE}
}{P \Rightarrow Q \vdash P \land Q \Rightarrow P}\;\mathtt{impI}
\qquad
\dfrac{
\dfrac{
\dfrac{
\dfrac{-}{P \vdash P}\;\mathtt{asm}
\quad
\dfrac{
\dfrac{
\dfrac{-}{P, Q \vdash P}\;\mathtt{asm}
}{Q, P \vdash P}\;\mathtt{rotate}
\quad
\dfrac{-}{Q, P \vdash Q}\;\mathtt{asm}
}{Q, P \vdash P \land Q}\;\mathtt{conjI}
}{P \Rightarrow Q, P \vdash P \land Q}\;\mathtt{impE}
}{P, P \Rightarrow Q \vdash P \land Q}\;\mathtt{rotate}
}{P \Rightarrow Q \vdash P \Rightarrow P \land Q}\;\mathtt{impI}
}{P \Rightarrow Q \vdash P \land Q \Leftrightarrow P}\;\mathtt{iffI}
$$

Example: Rule of Subsumption

$$
\dfrac{
\dfrac{
\dfrac{P, Q, P \Rightarrow Q \vdash P}{P \wedge Q, P \Rightarrow Q \vdash P}\,\mathtt{conjE}
}{P \Rightarrow Q \vdash P \wedge Q \Rightarrow P}\,\mathtt{impI}
\qquad
\dfrac{
\dfrac{
\dfrac{
\dfrac{-}{P \vdash P}\,\mathtt{asm}
\quad
\dfrac{
\dfrac{\dfrac{-}{P, Q \vdash P}\,\mathtt{asm}}{Q, P \vdash P}\,\mathtt{rotate}
\quad
\dfrac{-}{Q, P \vdash Q}\,\mathtt{asm}
}{Q, P \vdash P \wedge Q}\,\mathtt{conjI}
}{P \Rightarrow Q, P \vdash P \wedge Q}\,\mathtt{impE}
}{P, P \Rightarrow Q \vdash P \wedge Q}\,\mathtt{rotate}
}{P \Rightarrow Q \vdash P \Rightarrow P \wedge Q}\,\mathtt{impI}
}{P \Rightarrow Q \vdash P \wedge Q \Leftrightarrow P}\,\mathtt{iffI}
$$

Example: Rule of Subsumption

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$P, Q, P \Rightarrow Q \vdash P$}
\RightLabel{$\texttt{conjE}$}
\UnaryInfC{$P \wedge Q, P \Rightarrow Q \vdash P$}
\RightLabel{$\texttt{impI}$}
\UnaryInfC{$P \Rightarrow Q \vdash P \wedge Q \Rightarrow P$}
\AxiomC{$-$}
\RightLabel{$\texttt{asm}$}
\UnaryInfC{$P \vdash P$}
\AxiomC{$-$}
\RightLabel{$\texttt{asm}$}
\UnaryInfC{$P, Q \vdash P$}
\RightLabel{$\texttt{rotate}$}
\UnaryInfC{$Q, P \vdash P$}
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\RightLabel{$\texttt{asm}$}
\UnaryInfC{$Q, P \vdash Q$}
\RightLabel{$\texttt{conjI}$}
\BinaryInfC{$Q, P \vdash P \wedge Q$}
\RightLabel{$\texttt{impE}$}
\BinaryInfC{$P \Rightarrow Q, P \vdash P \wedge Q$}
\RightLabel{$\texttt{rotate}$}
\UnaryInfC{$P, P \Rightarrow Q \vdash P \wedge Q$}
\RightLabel{$\texttt{impI}$}
\UnaryInfC{$P \Rightarrow Q \vdash P \Rightarrow P \wedge Q$}
\RightLabel{$\texttt{iffI}$}
\BinaryInfC{$P \Rightarrow Q \vdash P \wedge Q \Leftrightarrow P$}
\end{prooftree}
$$

Example: Rule of Subsumption

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{\texttt{asm}}
\UnaryInfC{$P, Q, P \Rightarrow Q \vdash P$}
\RightLabel{\texttt{conjE}}
\UnaryInfC{$P \wedge Q, P \Rightarrow Q \vdash P$}
\RightLabel{\texttt{impI}}
\UnaryInfC{$P \Rightarrow Q \vdash P \wedge Q \Rightarrow P$}
\AxiomC{$-$}
\RightLabel{\texttt{asm}}
\UnaryInfC{$P \vdash P$}
\AxiomC{$-$}
\RightLabel{\texttt{asm}}
\UnaryInfC{$P, Q \vdash P$}
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\end{prooftree}
$$

Example: Rule of Subsumption

$$
\dfrac{
\dfrac{\dfrac{\dfrac{-}{P, Q, P \Rightarrow Q \vdash P}\;\texttt{asm}}{P \wedge Q, P \Rightarrow Q \vdash P}\;\texttt{conjE}}{P \Rightarrow Q \vdash P \wedge Q \Rightarrow P}\;\texttt{impI}
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{-}{P \vdash P}\;\texttt{asm} \quad \dfrac{\dfrac{\dfrac{-}{P, Q \vdash P}\;\texttt{asm}}{Q, P \vdash P}\;\texttt{rotate} \quad \dfrac{-}{Q, P \vdash Q}\;\texttt{asm}}{Q, P \vdash P \wedge Q}\;\texttt{conjI}}{P \Rightarrow Q, P \vdash P \wedge Q}\;\texttt{impE}}{P, P \Rightarrow Q \vdash P \wedge Q}\;\texttt{rotate}}{P \Rightarrow Q \vdash P \Rightarrow P \wedge Q}\;\texttt{impI}
}{P \Rightarrow Q \vdash P \wedge Q \Leftrightarrow P}\;\texttt{iffI}
$$

Outline

Negation

- The rule for negation introduction is used to prove $\Gamma \vdash \neg\, P$.
- The proof proceeds by assuming P and then showing a contradiction:

$$\frac{P, \Gamma \vdash \mathit{false}}{\Gamma \vdash \neg\, P}\ \texttt{notI}$$

- The rule for classical contradiction mirrors this and is used to prove $\Gamma \vdash P$.
- The proof proceeds by assuming $\neg\, P$ and then showing a contradiction:

$$\frac{\neg\, P, \Gamma \vdash \mathit{false}}{\Gamma \vdash P}\ \texttt{ccontr}$$

- Negation elimination is used with a hypothesis $\neg\, P$.
- The proof proceeds by proving the contradiction P:

$$\frac{\Gamma \vdash P}{\neg\, P, \Gamma \vdash R}\ \texttt{notE}$$

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Example: Proof with Negation

- Consider the sequent $\neg (P \vee Q) \vdash \neg Q$.
- We can use the negation introduction and elimination rules to prove this.
- Proof:
- We can use this argument twice to prove one of De Morgan's Laws.

Example: Proof with Negation

- Consider the sequent $\neg (P \vee Q) \vdash \neg Q$.
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Example: Proof with Negation

- Consider the sequent $\neg (P \vee Q) \vdash \neg Q$.
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- Proof:

$$\neg (P \vee Q) \vdash \neg Q$$

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Example: Proof with Negation

- Consider the sequent $\neg\,(P \vee Q) \vdash \neg\, Q$.
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$$\frac{}{\neg\,(P \vee Q) \vdash \neg\, Q}$$

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Example: Proof with Negation

- Consider the sequent $\neg\,(P \lor Q) \vdash \neg\, Q$.
- We can use the negation introduction and elimination rules to prove this.
- Proof:

$$\frac{}{\neg\,(P \lor Q) \vdash \neg\, Q}\ \texttt{notI}$$

- We can use this argument twice to prove one of De Morgan's Laws.

Example: Proof with Negation

- Consider the sequent $\neg (P \vee Q) \vdash \neg Q$.
- We can use the negation introduction and elimination rules to prove this.
- Proof:

$$\frac{Q, \neg (P \vee Q) \vdash \mathit{false}}{\neg (P \vee Q) \vdash \neg Q}\ \texttt{notI}$$

- We can use this argument twice to prove one of De Morgan's Laws.

Example: Proof with Negation

- Consider the sequent $\neg (P \lor Q) \vdash \neg Q$.
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$$\dfrac{\overline{Q, \neg (P \lor Q) \vdash \mathit{false}}}{\neg (P \lor Q) \vdash \neg Q}\ \texttt{notI}$$

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- Consider the sequent $\neg\,(P \vee Q) \vdash \neg\, Q$.
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$$\dfrac{\dfrac{\neg\,(P \vee Q), Q \vdash \mathit{false}}{Q, \neg\,(P \vee Q) \vdash \mathit{false}}\;\texttt{rotate}}{\neg\,(P \vee Q) \vdash \neg\, Q}\;\texttt{notI}$$

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$$
\cfrac{\cfrac{\cfrac{}{\neg (P \lor Q), Q \vdash \mathit{false}}\;\texttt{notE}}{Q, \neg (P \lor Q) \vdash \mathit{false}}\;\texttt{rotate}}{\neg (P \lor Q) \vdash \neg Q}\;\texttt{notI}
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$$\dfrac{\dfrac{\dfrac{\dfrac{}{Q \vdash P \lor Q}\ \texttt{disjI2}}{\neg (P \lor Q), Q \vdash \mathit{false}}\ \texttt{notE}}{Q, \neg (P \lor Q) \vdash \mathit{false}}\ \texttt{rotate}}{\neg (P \lor Q) \vdash \neg Q}\ \texttt{notI}$$

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$$
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$$
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$$

- We can use this argument twice to prove one of De Morgan's Laws.

Example: De Morgan's Law

Example: De Morgan's Law

$$\neg (P \vee Q) \vdash \neg P \wedge \neg Q$$

Example: De Morgan's Law

$$\frac{}{\neg\,(P \vee Q) \vdash \neg\,P \wedge \neg\,Q}$$

Example: De Morgan's Law

$$\dfrac{}{\neg\,(P \vee Q) \vdash \neg\, P \wedge \neg\, Q}\;\texttt{conjI}$$

Example: De Morgan's Law

$$\frac{\neg\,(P \lor Q) \vdash \neg\,P \qquad \neg\,(P \lor Q) \vdash \neg\,Q}{\neg\,(P \lor Q) \vdash \neg\,P \land \neg\,Q}\ \texttt{conjI}$$

Example: De Morgan's Law

$$\dfrac{\neg (P \vee Q) \vdash \neg P \qquad \dfrac{}{\neg (P \vee Q) \vdash \neg Q}}{\neg (P \vee Q) \vdash \neg P \wedge \neg Q}\ \texttt{conjI}$$

Example: De Morgan's Law

$$\dfrac{\neg\,(P \lor Q) \vdash \neg\,P \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{-}{Q \vdash Q}\;\texttt{asm}}{Q \vdash P \lor Q}\;\texttt{disjI2}}{\neg\,(P \lor Q), Q \vdash \mathit{false}}\;\texttt{notE}}{Q, \neg\,(P \lor Q) \vdash \mathit{false}}\;\texttt{rotate}}{\neg\,(P \lor Q) \vdash \neg\,Q}\;\texttt{notI}}{\neg\,(P \lor Q) \vdash \neg\,P \land \neg\,Q}\;\texttt{conjI}$$

Example: De Morgan's Law

$$
\dfrac{
\dfrac{}{\neg\,(P \vee Q) \vdash \neg\,P}
\qquad
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{Q \vdash Q}\;\texttt{asm}
}{Q \vdash P \vee Q}\;\texttt{disjI2}
}{\neg\,(P \vee Q), Q \vdash \mathit{false}}\;\texttt{notE}
}{Q, \neg\,(P \vee Q) \vdash \mathit{false}}\;\texttt{rotate}
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$$

Example: De Morgan's Law

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{-}{P \vdash P}\,\texttt{asm}}{P \vdash P \lor Q}\,\texttt{disjI1}}{\neg\,(P \lor Q), P \vdash \mathit{false}}\,\texttt{notE}}{P, \neg\,(P \lor Q) \vdash \mathit{false}}\,\texttt{rotate}}{\neg\,(P \lor Q) \vdash \neg\,P}\,\texttt{notI} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{-}{Q \vdash Q}\,\texttt{asm}}{Q \vdash P \lor Q}\,\texttt{disjI2}}{\neg\,(P \lor Q), Q \vdash \mathit{false}}\,\texttt{notE}}{Q, \neg\,(P \lor Q) \vdash \mathit{false}}\,\texttt{rotate}}{\neg\,(P \lor Q) \vdash \neg\,Q}\,\texttt{notI}}{\neg\,(P \lor Q) \vdash \neg\,P \land \neg\,Q}\,\texttt{conjI}$$

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Soundness and Completeness

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  An axiom is an inference rule with empty premises.

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  A derivation is tree of propositions labelled by rule names.

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  A derivation is a proof if all its leaves are axioms.

- Theorem (Soundness)

  The root sequent of a proof is valid.

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  If a sequent is valid, then there is a proof with that sequent as root.

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Outline

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